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On the stationary solutions of the full compressible Navier–Stokes equations and its stability with respect to initial disturbance ☆

Jianzhen Qian [a,b,*], Hui Yin [a,b]

[a] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China*
[b] *Graduate School of the Chinese Academy of Sciences, Beijing 100039, China*

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Abstract

This paper is concerned with the existence, uniqueness and nonlinear stability of stationary solutions to the Cauchy problem of the full compressible Navier–Stokes equations effected by external force of general form in $\mathbb{R}^3$.
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Contents

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* Corresponding author.
E-mail address: jzq03@126.com (J. Qian).

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1. Introduction and the statement of the main results

This paper is concerned with the nonlinear stability of stationary solutions to Cauchy problem of the full compressible Navier–Stokes equations

$$\begin{cases} \rho_t + \nabla\cdot(\rho v) = G, \\ v_t + (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v) - \dfrac{\nabla P}{\rho} + F, \\ \theta_t + (v\cdot\nabla)\theta + \dfrac{\theta P_\theta}{\rho c_\nabla}\nabla\cdot v = \dfrac{1}{\rho c_\nabla}\big\{\kappa\Delta\theta + \Psi(v) + H\big\}, \end{cases} \tag{1.1}$$

with initial data

$$(\rho, u, \theta)(t,x)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \to (\bar\rho, 0, \bar\theta) \quad \text{as } |x| \to +\infty. \tag{1.2}$$

Here $t \geqslant 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; $\rho(t,x) > 0$, $v = (v^1(t,x), v^2(t,x), v^3(t,x))$, $\theta(t,x)$ denote the density, velocity and temperature, respectively, P is the pressure; μ and μ' are the viscosity coefficients, κ the coefficient of the heat conduction, c_∇ the heat capacity at the constant volume; $F(x) = (F^1(x), F^2(x), F^3(x))$, $G(x)$, $H(x)$ are the given external force, mass source and energy source, respectively. Moreover, $\Psi = \Psi(v)$ is the dissipation function:

$$\Psi(v) = \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 + \mu'\sum_{j=1}^{3}\left(\frac{\partial v_j}{\partial x_j}\right)^2.$$

Our basic assumptions are as follows: $\bar\rho$, $\bar\theta$, κ, μ, μ' are the constants satisfying $\bar\rho, \bar\theta, \kappa, \mu > 0$, and $\frac{2}{3}\mu + \mu' \geqslant 0$; $c_\nabla(\rho,\theta)$, $P(\rho,\theta)$ are smooth functions of $\rho, \theta > 0$ satisfying $c_\nabla(\rho,\theta)$, $P(\rho,\theta)$, $P_\rho(\rho,\theta)$, $P_\theta(\rho,\theta) > 0$.

As a fundamental equation in the fluid mechanics, (1.1), (1.2) has been studied by many mathematicians and a lot of results have been obtained (a complete list of references in this direction is beyond the scope of this manuscript, the interested reader is referred, however, to the monographs [7,15] and the references cited therein). In the following, we only review some results closely related to the theme of the manuscript: For the case $G = F = H \equiv 0$, Matsumura and Nishida [9,10] first proved the stability of a constant state $(\bar\rho, 0, \bar\theta)$ in H^3-framework with respect to small initial disturbance. When the external force is the potential force, i.e., $F = -\nabla\Phi$, where Φ is a scalar function, and $G = H = 0$, the stationary solution (ρ^*, v^*, θ^*) to (1.1) in the neighborhood of $(\bar\rho, 0, \bar\theta)$ in $\mathcal{H}^{2,2,2}$ has the form

$$\int_{\bar\rho}^{\rho^*(x)} \frac{P_\rho(\eta, \bar\theta)}{\eta}\,d\eta + \Phi(x) = 0, \quad v^*(x) = 0, \quad \theta^* = \bar\theta.$$

For this case, Matsumura and Nishida [11] proved the stability of $(\rho^*(x), 0, \bar{\theta})$ in H^3-framework with respect to small initial disturbance in an exterior domain. When the external force is given by the general form $F = \nabla \cdot F_1 + F_2$ and also mass source G appears, the stationary solution is nontrivial in general. For this problem, when F is small in certain norm and $G = H = 0$, the existence and uniqueness of stationary solution in an exterior domain is studied by Matsumura and Nishida [12]. Then for the isentropic flows, the existence and nonlinear stability of stationary solutions have been studied by Shibata and Tanaka [17,18] in $\mathbb{R}^3$, and also by Novotny and Padula [13,14] and Tanaka [19] in an exterior domain of $\mathbb{R}^3$ with $G = 0$.

In this paper, as a continuation of [17], we will study this problem for the non-isentropic flows in whole space $\mathbb{R}^3$. Our aim is to study the existence and uniqueness of the stationary solution, which is a small strong solution in the neighborhood of $(\bar{\rho}, 0, \bar{\theta})$ to the full compressible Navier–Stokes equations (1.1), and also its stability with respect to small initial perturbation.

Before stating our main results, we shall explain our notations first, which are borrowed from [17]. For convenience of readers, we also list them as follows.

Notations. In the rest of this paper, we use the standard notation in vector analysis. For example, we put for scalar u, vectors $v = (v^1, v^2, v^3)$, $w = (w^1, w^2, w^3)$ and matrix $f = (f^{ij})_{1\leqslant i,j\leqslant 3}$

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, \qquad \Delta v = \left(\Delta v^1, \Delta v^2, \Delta v^3\right), \qquad (v\cdot\nabla)u = \sum_{i=1}^{3} v^i \frac{\partial u}{\partial x_i},$$

$$(v\cdot\nabla)w = \left((v\cdot\nabla)w^1, (v\cdot\nabla)w^2, (v\cdot\nabla)w^3\right),$$

$$\nabla^k u = \left(\partial_x^\alpha u \;\middle|\; |\alpha| = k\right), \qquad \nabla^k v = \left(\partial_x^\alpha v_i \;\middle|\; |\alpha| = k,\ i = 1,2,3\right),$$

$$\nabla_i u = \frac{\partial u}{\partial x_i}, \qquad \nabla\cdot v = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x_i}, \qquad \nabla\cdot f = \left(\sum_{j=1}^{3} \frac{\partial f^{1j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial f^{2j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial f^{3j}}{\partial x_j}\right).$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$.

Next, we introduce some function spaces that will be used later, put for scalars u_1, u_2 and vectors $v = (v^1, \dots, v^n)$, $w = (w^1, \dots, w^n)$,

$$\|u_1\|_{L_p} = \left(\int_{\mathbb{R}^3} \left|u_1(x)\right|^p dx\right)^{\frac{1}{p}}, \qquad \|v\|_{L_p} = \left(\sum_{i=1}^{n} \left\|v^i\right\|_{L_p}^p\right)^{\frac{1}{p}} \quad (1 \leqslant p < \infty),$$

$$\|u_1\|_{L_\infty} = \sup_{\mathbb{R}^3}\left|u_1(x)\right|, \qquad \|v\|_{L_\infty} = \max_{1\leqslant i\leqslant n}\left\|v^i(x)\right\|_{L_\infty}, \qquad (u_1, u_2) = \int_{\mathbb{R}^3} u_1 u_2\, dx,$$

$$(v, w) = \sum_{i=1}^{n} \left\langle v^i, w^i\right\rangle, \qquad \|v\|_k = \left(\sum_{0\leqslant \nu\leqslant k} \left\|\nabla^\nu v\right\|^2\right)^{\frac{1}{2}}, \qquad \|\cdot\| = \|\cdot\|_{L_2}.$$

Let L_p denote the usual L_p space, and

$$H^k = \left\{u \in L_{1,\mathrm{loc}} \;\middle|\; \|u\|_k < \infty\right\}, \qquad \hat{H}^k = \left\{u \in L_{1,\mathrm{loc}} \;\middle|\; \nabla u \in H^{k-1}\right\},$$

where u is either a vector or scalar. Further we put

$$\mathcal{H}^{k,l} = \{(q,v) \mid q \in H^k,\ v \in H^l\}, \qquad \hat{\mathcal{H}}^{k,l} = \{(q,v) \mid q \in \hat{H}^k,\ v \in \hat{H}^l\},$$
$$\mathcal{H}^{j,k,l} = \{(q,v,\vartheta) \mid q \in H^j,\ v \in H^k,\ \vartheta \in H^l\},$$
$$\hat{\mathcal{H}}^{j,k,l} = \{(q,v,\vartheta) \mid q \in \hat{H}^j,\ v \in \hat{H}^k,\ \vartheta \in \hat{H}^l\},$$

and

$$\|(q,v)\|_{k,l} = \|q\|_k + \|v\|_l, \qquad \|(q,v,\vartheta)\|_{j,k,l} = \|q\|_j + \|v\|_k + \|\vartheta\|_l.$$

Definition 1.1.

$$I_\epsilon^k = \{q \in H^k \mid \|q\|_{I^k} < \epsilon\}, \qquad J_\epsilon^k = \{v \in H^k \mid \|v\|_{J^k} < \epsilon\}, \qquad j_\epsilon^k = \{\vartheta \in H^k \mid \|v\|_{j^k} < \epsilon\},$$

where

$$\|q\|_{I^k} = \begin{cases} \|q\|_{\hat{I}^k} + \|(1+|x|)^2\nabla q\|_{L_\infty}, & \text{if } k \geqslant 4, \\ \|q\|_{\hat{I}^k}, & \text{otherwise,} \end{cases}$$

and $\|\cdot\|_{\hat{I}^k}$ is defined by

$$\|u\|_{\hat{I}^k} = \|u\|_{L_6} + \sum_{\nu=1}^{k} \|(1+|x|)^\nu \nabla^\nu u\| + \|(1+|x|)^2 u\|_{L_\infty};$$

$$\|v\|_{J^k} = \begin{cases} \|v\|_{\hat{J}^k} + \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty} + \|(1+|x|)^2\nabla^2 v\|_{L_\infty}, & \text{if } k \geqslant 5, \\ \|v\|_{\hat{J}^k} + \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty}, & \text{otherwise,} \end{cases}$$

where $\|\cdot\|_{\hat{J}^k}$ is defined by $\|u\|_{\hat{J}^k} = \|u\|_{L_6} + \sum_{\nu=1}^{k} \|(1+|x|)^{\nu-1}\nabla^\nu u\|$;

$$\|\vartheta\|_{j^k} = \begin{cases} \|\vartheta\|_{\hat{J}^k} + \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+1}\nabla^\nu \vartheta\|_{L_\infty} + \|(1+|x|)^2\nabla^2 \vartheta\|_{L_\infty}, & \text{if } k \geqslant 5, \\ \|\vartheta\|_{\hat{J}^k} + \sum_{\nu=0}^{2} \|(1+|x|)^{\nu}\nabla^\nu \vartheta\|_{L_\infty}, & \text{otherwise.} \end{cases}$$

Moreover, we put

$$\Lambda_\epsilon^{j,k,l} = \{(q,v,\vartheta) \mid q \in I_\epsilon^j, v \in J_\epsilon^k,\ \vartheta \in j_\epsilon^l\},$$
$$\dot{\Lambda}_\epsilon^{j,k,l} = \{(q,v,\vartheta) \in \Lambda_\epsilon^{j,k,l} \mid \nabla\cdot v = \nabla\cdot V_1 + V_2,\ \text{for some } V_1, V_2,$$
$$\text{such that } \|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1} V_2\|_{L_1} \leqslant \epsilon\},$$
$$\|(q,v,\theta)\|_{\Lambda^{j,k,l}} = \|q\|_{I^k} + \|v\|_{J^k} + \|\vartheta\|_{j^l},$$

$$\mathcal{L} = \{U \mid U = \nabla\cdot U_1 + U_2 \text{ for some } U_1, U_2,\ \text{and satisfies } \|U\|_{\mathcal{L}} < \infty\},$$

where

$$\|U\|_{\mathcal{L}}=\sum_{\nu=1}^{3}\big\|\big(1+|x|\big)^{\nu+1}\nabla^{\nu}U\big\|+\big\|\big(1+|x|\big)^{3}(U,\nabla U)\big\|_{L_\infty}+\big\|\big(1+|x|\big)^{2}U_1\big\|_{L_\infty}+\|U_2\|_{L_1}.$$

In this paper, we consider the case where the mass source G, the external force F and energy source H are given by the following form

$$\begin{pmatrix} G\\ F\\ H\end{pmatrix}=\nabla\cdot\begin{pmatrix} G_1\\ F_1\\ H_1\end{pmatrix}+\begin{pmatrix} G_2\\ F_2\\ H_2\end{pmatrix},$$

where $F_1=(F_1^{ij}(x))_{1\leqslant i,j\leqslant 3}$, $F_2=(F_2^{i}(x))_{1\leqslant i\leqslant 3}$; $G_1=(G_1^{i}(x))_{1\leqslant i\leqslant 3}$, $G_2=G_2(x)$; $H_1=(H_1^{i}(x))_{1\leqslant i\leqslant 3}$, $H_2=H_2(x)$.

Now we begin to state the main result of this paper. First, we consider the stationary problem corresponding to the initial one (1.1)–(1.2):

$$\begin{cases} \nabla\cdot(\rho v)=G,\\ (v\cdot\nabla)v=\dfrac{\mu}{\rho}\Delta v+\dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v)-\dfrac{\nabla P}{\rho}+F,\\ (v\cdot\nabla)\theta+\dfrac{\theta P_\theta}{\rho c_\nabla}\nabla\cdot v=\dfrac{1}{\rho c_\nabla}\big\{\kappa\Delta\theta+\Psi(v)+H\big\}. \end{cases}\tag{1.3}$$

Regarding ρ as a smooth function of (P,θ), our first theorem is concerned with the existence of the stationary solution (P,v,θ) of the stationary problem (1.3), and its weighted L_2 and L_∞ estimates.

Theorem 1.1. *Let $\bar\rho$, $\bar\theta$ be any positive constants, and set $\bar P=P(\bar\rho,\bar\theta)$. There exist small constants $c_0>0$ and $\epsilon_0>0$ depending on $\bar\rho$ and $\bar\theta$, such that if $(G,F,H)\in\mathcal{H}^{4,3,3}$ and satisfies the estimate*:

$$\big\|(G,F,H)\big\|_{\mathcal{L}}+\big\|\big(1+|x|\big)^{4}\nabla^{4}G\big\|+\big\|\big(1+|x|\big)^{-1}G\big\|_{L_1}\leqslant c_0\epsilon$$

for some $\epsilon\leqslant\epsilon_0$, then (1.3) *admits a solution of the form*: $(P,v,\theta)=(\bar P+q,v,\bar\theta+\vartheta)$ *where $(q,v,\vartheta)\in\dot\Lambda_{\epsilon}^{4,5,5}$. Furthermore the solution is unique in the following sense*: *if there is another solution $(\bar P+q_1,v_1,\bar\theta+\vartheta_1)$ satisfying* (1.3) *with the same (G,F,H), and $\|(q_1,v_1,\vartheta_1)\|_{\Lambda^{4,5,5}}\leqslant\epsilon$, then $(q_1,v_1,\vartheta_1)=(q,v,\vartheta)$.*

Next, we consider the stability of the stationary solution of (1.3) with respect to the initial disturbance. Denote by (ρ^*,v^*,θ^*) the stationary solution obtained in Theorem 1.1, then the stability of (ρ^*,v^*,θ^*) means the solvability of the non-stationary problem (1.1) with initial data (1.2). Let us introduce first the class of functions which solutions of (1.1)–(1.2) belong to.

Definition 1.2.

$$\mathcal{C}\big(0,T;\mathcal{H}^{j,k,l}\big)=\big\{(\sigma,w,\theta)\;\big|\;\sigma(t,x)\in C^0\big(0,T;H^j\big)\cap C^1\big(0,T;H^{j-1}\big),\\ w(t,x)\in C^0\big(0,T;H^k\big)\cap C^1\big(0,T;H^{k-2}\big),\\ \theta(t,x)\in C^0\big(0,T;H^l\big)\cap C^1\big(0,T;H^{l-2}\big)\big\}.$$

Then we have the following theorem.

Theorem 1.2. *There exist $C>0$ and $\delta>0$ such that if $\|(\rho_0-\rho^*,v_0-v^*,\theta_0-\theta^*)\|_{3,3,3}\leqslant\delta$, then* (1.1) *admits a unique solution $(\rho,v,\theta)=(\rho^*+\sigma,v^*+w,\theta^*+\vartheta)$ globally in time, where $(\sigma,w,\vartheta)\in\mathcal{C}(0,\infty;\mathcal{H}^{3,3,3})$, $\nabla\sigma,w_t,\vartheta_t\in L_2(0,\infty;H^2)$, $\nabla w,\nabla\vartheta\in L_2(0,\infty;H^3)$. Moreover, the solution (σ,w,ϑ) satisfies the estimate*:

$$\big\|(\sigma,w,\vartheta)(t)\big\|_{3,3,3}^2+\int_0^t\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(s)\big\|_{2,3,3}^2+\big\|(w_t,\vartheta_t)(s)\big\|_{2,2}^2\,ds\\ \leqslant C\big\|(\sigma,w,\vartheta)(0)\big\|_{3,3,3}^2.$$

Remark 1.1. Theorem 1.2 implies the nonlinear stability of the stationary solution (ρ^*,v^*,θ^*) in the H^3-framework with respect to small initial disturbance.

Before concluding this section, we outline the main ideas we used to prove our main results. Note that in [12], for an exterior domain problem, Matsumura and Nishida used a regularization method to deduce the existence of stationary solution to (1.1), while in $\mathbb{R}^3$, the techniques developed by Shibata and Tanaka [17] to study the isentropic case is based on a corresponding linear theory in the L_2-framework by employing the Banach closed range theorem and some delicate energy estimates. In their analysis, the suitably chosen weights in the weighted L_2 and L_∞ estimates for the linearized equations play an important role when applying the linear theory to the corresponding nonlinear problem. Since our main purpose of this paper is to generalize the results in [17] on the isentropic flow to the full compressible Navier–Stokes equations, certainly we hope that the argument developed in [17] can still be used to treat the full compressible Navier–Stokes equations. Indeed, by employing the techniques used in [17] directly, if we choose (ρ,v,θ) as independent variables, we can deduce some similar estimates for the solutions of the corresponding linearized equations, from which and the Banach closed range theorem, we can deduce a similar existence results for the linearized equations. To use the linear theory to deduce a global existence result for the nonlinear problem, as in [17], one need to get certain weighted L_2 and L_∞ estimates for the linearized equations. For the isentropic flow, the weighted L_2 estimate is of the form $\sum_{\nu=1}^{\ell}\|(1+|x|)^\nu(\nabla^\nu\sigma,\nabla^{\nu+1}v)\|$. To deduce such an estimate, the fact that $\|(1+|x|)^\ell\nabla^\ell\sigma\|$ can be controlled by $\|(1+|x|)^\ell\nabla^{\ell+1}v\|$, which, for the isentropic flow, is an easy consequence of the momentum equation for the linearized equations (cf. the estimate immediately after (2.30) of [17]), plays an important role in their analysis. We note, however, that for the non-isentropic flow, due to the appearance of the extra term $q_\theta\nabla\theta$ in the momentum equation, we cannot hope to deduce some similar weighted L_2 and L_∞ estimates on σ for the non-isentropic case. But, as mentioned before, such a type of estimates plays an essential role in deducing the global existence results for the nonlinear equations. To overcome this, we choose (P, v, θ) as independent variables. Based on this and by combining the weighted L_2 estimates and the L_∞ spatial decay estimates as in [17], we can indeed obtained the existence of the stationary solution (P^*, v^*, θ^*) in proper function space. As to the stability of the stationary solutions obtained above, for an exterior domain problem, Matsumura and Nishida [12] mentioned a possible way but they did not give the details, while for the Cauchy problem of the isentropic flow, Shibata and Tanaka [17] give a proof of the stability of the above stationary solutions. As for the full compressible Navier–Stokes equations, based on the properties we obtained on the stationary solutions, we can also deduce the corresponding nonlinear stability results. It is worth to pointing out that, for the non-isentropic flow, although we can deduce a weighted L_2 estimate like $\sum_{\nu=1}^{\ell} \|(1+|x|)^\nu (\nabla^\nu q^*, \nabla^{\nu+1} v^*, \nabla^{\nu+1}\vartheta^*)\|$ for the stationary solutions, from which we can deduce an estimate on $\sum_{\nu=1}^{\ell} \|(1+|x|)^{\nu-1}\nabla^\nu \sigma^*\|$ while for the isentropic case, the corresponding estimate obtained in [17] is an estimate on $\sum_{\nu=1}^{\ell} \|(1+|x|)^\nu \nabla^\nu \sigma^*\|$. Although for the non-isentropic case, the properties we obtained on the stationary solutions are not so good as for the isentropic case, these estimates together with some delicate estimates are enough to deduce the desired nonlinear stability result of the stationary solutions.

Another interesting problem is on the convergence rate of non-stationary solutions to the stationary solutions when the time variables goes to infinity, we note that there has been many works for the cases $G = F = H = 0$ or $F = -\nabla\Phi$ and $G = H = 0$, or for general F and G but for the isentropic flows (cf. [1–6,8,16,18,20,21] and the reference cited therein). But to obtain the convergence rate in our case, it appears to be more delicate since the stationary solution is nontrivial generally. We will consider this problem in a forthcoming paper.

The rest of the paper is arranged as follows. The stationary problem is studied in Section 2, while the non-stationary problem is studied in Section 3.

2. Stationary problem

In this section, we study the stationary problem (1.3). As mentioned in Section 1, by regarding ρ as the function of (P, θ), changing the variables as $(P, v, \theta) \to (\bar{P}+q, v, \bar{\theta}+\vartheta)$, and rewriting the third equation by using the first one, (1.3) can be then reduced to the following equation:

$$\begin{cases} \nabla\cdot v + \dfrac{\rho_P}{\rho}(v\cdot\nabla)q = -\dfrac{\rho_\theta}{\rho} v\cdot\nabla\vartheta + \dfrac{G}{\rho}, \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \nabla q = -\rho(v\cdot\nabla)v + \rho F, \\ -\kappa\Delta\vartheta = -\left(\rho c_\nabla + \dfrac{\theta\rho_\theta^2}{\rho\rho_P}\right)(v\cdot\nabla)\vartheta - \dfrac{\theta\rho_\theta}{\rho}(v\cdot\nabla)q + \Psi(v) + \dfrac{\theta\rho_\theta}{\rho\rho_P}G + H. \end{cases} \tag{2.1}$$

In the rest of this section, we will focus on Eqs. (2.1). To this end, as in [17], we need first to study the corresponding linearized problem which will play an important role in proving Theorem 1.1.

2.1. Weighted L_2 theory for linearized problem

Firstly, we consider the linearized equation of (2.1)

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)q = g, \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \nabla q = f, \\ -\kappa\Delta\vartheta = h. \end{cases} \tag{2.2}$$

Here $a = (a^1(x), a^2(x), a^3(x))$, $(g, h, f) \in \mathcal{H}^{k,k-1,k-1}$ are given. Throughout this subsection, we assume that

$$a \in \hat{H}^k, \quad \big\|\big(1+|x|\big)a\big\|_{L_\infty} + \sum_{\nu=1}^{k} \big\|\big(1+|x|\big)^{\nu-1}\nabla^\nu a\big\| \leqslant \delta. \tag{2.3}$$

We note that the system of equations $(2.2)_1$–$(2.2)_2$, which are independent of ϑ, has been well studied by Shibata and Tanaka in [17], while $(2.2)_3$ is easy to be dealt with. Thus by repeating the argument in [17], we can get the following result for the problem (2.2).

Lemma 2.1. *Assume that $(g, f, h) \in \mathcal{H}^{k,k-1,k-1}$ for some $k \geqslant 2$ and $\||x|(g, f, h)\| < \infty$. Then there exists $\delta_0(\mu, \mu') > 0$ such that if δ in* (2.3) *satisfies $\delta \leqslant \delta_0$,* (2.2) *admits a solution $(q, v, \vartheta) \in \hat{\mathcal{H}}^{k,k+1,k+1}$ which satisfies the estimate*:

$$\begin{aligned}
&\big\|(q, v, \vartheta)\big\|_{L_6} + \big\|(\nabla q, \nabla v, \nabla\vartheta)\big\|_{k-1,k,k} \\
&\quad \leqslant C\big\{\big\|\big(1+|x|\big)(g, f, h)\big\| + \big\|(\nabla g, \nabla f, \nabla h)\big\|_{k-1,k-2,k-2}\big\},
\end{aligned} \tag{2.4}$$

where C is a constant depending only on μ, μ' and κ.

In the rest of this subsection we fix k to be 3 or 4 and put

$$f = -b_1 \cdot \nabla c_1 + \tilde{f}, \qquad h = -b_2 \cdot \nabla c_2 + \tilde{h}.$$

Under the assumption that

$$(g, \tilde{f}, \tilde{h}) \in \mathcal{H}^{k,k-1,k-1}, \quad b_1, b_2, c_1 \in J^{k+1}, \quad c_2 \in \dot{J}^{k+1}, \tag{2.5}$$

$$\big\|\big(1+|x|\big)g\big\| + \sum_{\nu=1}^{k}\big\|\big(1+|x|\big)^{\nu}\nabla^\nu g\big\| + \sum_{\nu=0}^{k-1}\big\|\big(1+|x|\big)^{\nu+1}\nabla^\nu(\tilde{f}, \tilde{h})\big\| \leqslant \infty, \tag{2.6}$$

we have the following weighted L_2 estimate for the solution to (2.2).

Lemma 2.2. *Let $(q, v, \theta) \in \hat{\mathcal{H}}^{k,k+1,k+1}$ be a solution to* (2.2) *satisfying* (2.4)*. Then there exists $\delta_0(\mu, \mu') > 0$ such that if δ in* (2.3) *satisfies $\delta \leqslant \delta_0$, for any integer with $1 \leqslant \ell \leqslant k$, we have the estimate*:

$$\begin{aligned}
&\sum_{\nu=1}^{\ell}\big\|\big(1+|x|\big)^{\nu}\big(\nabla^\nu q, \nabla^{\nu+1} v, \nabla^{\nu+1}\vartheta\big)\big\| \\
&\quad \leqslant C\Bigg\{\|\nabla v\| + \|\nabla\vartheta\| + \|b_1\|_{J^{k+1}}\|c_1\|_{J^{k+1}} + \|b_2\|_{J^{k+1}}\|c_2\|_{\dot{J}^{k+1}} \\
&\qquad + \sum_{\nu=1}^{\ell}\big\|\big(1+|x|\big)^{\nu}\big(\nabla^\nu g, \nabla^{\nu-1}\tilde{f}, \nabla^{\nu-1}\tilde{h}\big)\big\|\Bigg\},
\end{aligned} \tag{2.7}$$

where C is a constant depending only on μ, μ' and κ.

The proof of this lemma is similar to that of Lemma 2.3 in Shibata and Tanaka [17], and we omit the details for brevity.

Now combining Lemmas 2.1 and 2.2, we have the following theorem.

Theorem 2.1. *There exists $\delta_0 = \delta_0(\mu, \mu') > 0$ such that if δ in* (2.3) *satisfies $\delta \leqslant \delta_0$, then* (2.2) *admits a solution $(q, v, \vartheta) \in \hat{\mathcal{H}}^{k,k+1,k+1}$ which satisfies the estimate*:

$$\begin{aligned}
&\big\|(q, v, \vartheta)\big\|_{L_6} + \sum_{\nu=1}^{k} \big\| \big(1+|x|\big)^{\nu} \nabla^{\nu} q \big\| + \sum_{\nu=1}^{k+1} \big\| \big(1+|x|\big)^{\nu-1} \big(\nabla^{\nu} v, \nabla^{\nu} \vartheta\big) \big\| \\
&\quad \leqslant C \Bigg\{ \|b_1\|_{J^{k+1}} \|c_1\|_{J^{k+1}} + \|b_2\|_{J^{k+1}} \|c_2\|_{J^k} + \big\| \big(1+|x|\big) g \big\| \\
&\qquad + \sum_{\nu=1}^{k} \big\| \big(1+|x|\big)^{\nu} \nabla^{\nu} g \big\| + \sum_{\nu=0}^{k-1} \big\| \big(1+|x|\big)^{\nu+1} \nabla^{\nu} (\tilde{f}, \tilde{h}) \big\| \Bigg\},
\end{aligned} \tag{2.8}$$

where C is a constant depending only on μ, μ' and κ.

2.2. *Proof of Theorem 1.1*

In this subsection, we shall construct a solution to (1.3) by the contraction mapping principle in $\dot{\Lambda}_{\epsilon}^{4,5,5}$. To this end, we first consider the following iterated equations:

$$\begin{cases}
\nabla \cdot v + \dfrac{\tilde{\rho}_P}{\tilde{\rho}} (\tilde{v} \cdot \nabla) q = g, \\
-\mu \Delta v - (\mu + \mu') \nabla (\nabla \cdot v) + \nabla q = -\bar{\rho} (\tilde{v} \cdot \nabla) \tilde{v} + \tilde{f}, \\
-\kappa \Delta \vartheta = -\bar{\eta}_1 (\tilde{v} \cdot \nabla) \tilde{\vartheta} + \tilde{h},
\end{cases} \tag{2.9}$$

where

$$\begin{cases}
g = -\dfrac{\tilde{\rho}_\theta}{\tilde{\rho}} \tilde{v} \cdot \nabla \tilde{\vartheta} + \dfrac{G}{\tilde{\rho}}, \\
\tilde{f} = -(\tilde{\rho} - \bar{\rho})(\tilde{v} \cdot \nabla) \tilde{v} + \tilde{\rho} F, \\
\tilde{h} = -(\eta_1 - \bar{\eta}_1)(\tilde{v} \cdot \nabla) \tilde{\vartheta} - \eta_2 (\tilde{v} \cdot \nabla) \tilde{q} + \Psi(\tilde{v}) + \eta_3 G + H, \\
\eta_1 = \tilde{\rho} \tilde{c}_{\nabla} + \dfrac{\tilde{\theta} \tilde{\rho}_\theta^2}{\tilde{\rho} \tilde{\rho}_P}, \quad \eta_2 = \dfrac{\tilde{\theta} \tilde{\rho}_\theta}{\tilde{\rho}}, \quad \eta_3 = \dfrac{\tilde{\theta} \tilde{\rho}_\theta}{\tilde{\rho} \tilde{\rho}_P}.
\end{cases} \tag{2.10}$$

Here, $(\tilde{q}, \tilde{v}, \tilde{\vartheta})(x) \in \dot{\Lambda}_{\epsilon}^{4,5,5}$ is given, and $\tilde{\rho}_P = \rho_P(\bar{P} + \tilde{q}, \bar{\theta} + \tilde{\vartheta})$, $\bar{\eta}_1 = \eta_1(\bar{P}, \bar{\theta})$, etc.

2.2.1. *Introduction of solution map T for (2.1)*

Firstly, we apply Theorem 2.1 to (2.9) to get the weighted L_2 estimate. In this case, we have

$$\begin{cases}
a = \dfrac{\tilde{\rho}_P}{\tilde{\rho}} \tilde{v}, \quad b_1 = \bar{\rho} \tilde{v}, \quad c_1 = \tilde{v}; \\
b_2 = \bar{\eta}_1 \tilde{v}, \quad c_2 = \tilde{\vartheta},
\end{cases} \tag{2.11}$$

and g, $\tilde{f}$, $\tilde{h}$ in Theorem 2.1 are defined as in (2.10). We choose $\epsilon > 0$ sufficiently small such that $\frac{\bar{\rho}}{2} < \tilde{\rho} < 2\bar{\rho}$ and δ in (2.3) satisfy $\delta \leqslant \delta_0$ ($\delta_0 > 0$ is given by Theorem 2.1), as follows from the Sobolev inequality. Also we can check directly that (2.5) holds for $k = 4$. Moreover, we have

$$\big\|\big(1+|x|\big)g\big\| + \sum_{\nu=1}^{4}\big\|\big(1+|x|\big)^{\nu}\nabla^{\nu}g\big\| + \sum_{\nu=0}^{3}\big\|\big(1+|x|\big)^{\nu+1}\nabla^{\nu}(\tilde{f},\tilde{h})\big\| \leqslant C\big\{\epsilon^2 + K_0\big\}$$

for some constant $C = C(\bar{\rho}, \bar{\theta})$, where K_0 is defined by

$$K_0 = \big\|\big(1+|x|\big)^2(G,F,H)\big\|_{L_\infty} + \sum_{\nu=1}^{3}\big\|\big(1+|x|\big)^{\nu+1}\nabla^{\nu}(G,F,H)\big\| + \big\|\big(1+|x|\big)^4\nabla^4 G\big\|. \tag{2.12}$$

With the above observations in hand, we have by applying Theorem 2.1 with $k = 4$ to (2.9) that

Lemma 2.3. *Let* $(G, F, H) \in \mathscr{H}^{4,3,3}$ *satisfy* (2.12). *Then there exists* ϵ_0 *such that if* $\epsilon \leqslant \epsilon_0$, (2.12) *with* $(\tilde{q}, \tilde{v}, \tilde{\vartheta})(x) \in \dot{\Lambda}_\epsilon^{4,5,5}$ *has a solution* $(q, v, \vartheta) \in \hat{\mathscr{H}}^{4,5,5}$ *which satisfies the estimate*:

$$\big\|(q,v,\vartheta)\big\|_{L_6} + \sum_{\nu=1}^{4}\big\|\big(1+|x|\big)^{\nu}\nabla^{\nu}q\big\| + \sum_{\nu=1}^{4}\big\|\big(1+|x|\big)^{\nu}\big(\nabla^{\nu+1}v, \nabla^{\nu+1}\vartheta\big)\big\| \leqslant C\big\{\epsilon^2 + K_0\big\}, \tag{2.13}$$

where C depends only on $\bar{\rho}$, $\bar{\theta}$, μ, μ' and κ.

Based on Lemma 2.3, we can define the solution map $(\tilde{q}, \tilde{v}, \tilde{\vartheta}) \mapsto (q, v, \vartheta)$: $\dot{\Lambda}_\epsilon^{4,5,5} \to \hat{\mathscr{H}}^{4,5,5}$ by $(q, v, \vartheta) = T(\tilde{q}, \tilde{v}, \tilde{\vartheta})$.

Having defined the solution operator T, we will use the contraction mapping principle to prove Theorem 1.1. To this end, we first show that T maps $\dot{\Lambda}_\epsilon^{4,5,5}$ into itself. That is, we need to show that $(\tilde{q}, \tilde{v}, \tilde{\vartheta}) \in \dot{\Lambda}_\epsilon^{4,5,5}$ leads to $(q, v, \vartheta) \in \dot{\Lambda}_\epsilon^{4,5,5}$. For this purpose, we first give the following lemma which will play an important role when we estimate the solution by L_∞-norm.

Lemma 2.4. *Let $E(x)$ be a scalar function satisfying*

$$\big|\partial_x^\alpha E(x)\big| \leqslant \frac{C_\alpha}{|x|^{|\alpha|+1}}, \quad |\alpha| = 0, 1, 2.$$

(i) *If $\phi(x)$ is a smooth scalar function of the form $\phi = \nabla \cdot \phi_1 + \phi_2$ satisfying*

$$L_1(\phi) \equiv \big\|\big(1+|x|\big)^3\phi\big\|_{L_\infty} + \big\|\big(1+|x|\big)^2\phi_1\big\|_{L_\infty} + \|\phi_2\|_{L_1} < \infty,$$

then we have for any multi-index α with $|\alpha| = 0, 1$

$$\big|\partial_x^\alpha(E*\phi)(x)\big| \leqslant \frac{C_\alpha}{|x|^{|\alpha|+1}} L_1(\phi).$$

(ii) *If $\phi(x)$ is a smooth scalar function of the form $\phi = \phi_1\phi_2$ satisfying*

$$L_2(\phi) \equiv \big\|\big(1+|x|\big)^2\phi\big\|_{L_\infty} + \big\|\big(1+|x|\big)^3(\nabla\phi_1)\phi_2\big\|_{L_\infty} + \big\|\big(1+|x|\big)^3\phi_1(\nabla\phi_2)\big\|_1 < \infty,$$

then we have for any multi-index α with $|\alpha| = 1, 2$

$$\big|\partial_x^\alpha(E*\phi)(x)\big| \leqslant \frac{C_\alpha}{|x|^{|\alpha|}}L_2(\phi).$$

Here C_α denotes a constant depending only on α.

This lemma can be proved by partitioning $\mathbb{R}^3$ and estimating the integral in each domain of $\mathbb{R}^3$ respectively, cf. [13].

With the aid of the Helmholtz decomposition and the Fourier transform, the solution of (2.9) can be formulated as follows, cf. [17]:

$$v = \omega + \nabla p, \quad q = \Phi + (2\mu' + \mu)\Delta p, \quad \vartheta = E_0 * \Theta, \tag{2.14}$$

where

$$\begin{cases} \omega^j(x) = \displaystyle\sum_{i=1}^3 E^{ij} * f^i(x), \\ p(x) = E_0 * R(x), \\ \Phi(x) = \displaystyle\sum_{i=1}^3 \nabla_i\big(E_0 * f^i\big)(x) = E * (\nabla\cdot f), \end{cases} \tag{2.15}$$

and

$$\begin{cases} E^{ij}(x) = \dfrac{1}{8\pi\mu}\left(\dfrac{\delta^{ij}}{|x|} - \dfrac{x_ix_j}{|x|^3}\right), \quad E_0 = -\dfrac{1}{4\pi}|x|^{-1}, \\ f^i = -\bar\rho(\tilde v\cdot\nabla)\tilde v^i + \tilde f^i, \\ R = -\dfrac{\tilde\rho_P}{\tilde\rho}(\tilde v\cdot\nabla)q + g, \\ \Theta = \dfrac{1}{\bar\kappa}\big\{\bar\eta_1(\tilde v\cdot\nabla)\tilde\vartheta + \tilde h\big\}. \end{cases} \tag{2.16}$$

Now we shall estimate L_∞-norm of the solution to (2.9) by using Lemma 2.4.

Lemma 2.5. *Let $(G, F, H) \in \mathcal{H}^{4,3,3}$ satisfy the following estimate*:

$$\bar K \equiv \big\|(G, F, H)\big\|_{\mathcal{L}} + \big\|\big(1+|x|\big)^4\nabla^4 G\big\| < \infty.$$

If $(q, v, \vartheta) \in \hat{\mathcal{H}}^{4,5,5}$ is a solution to (2.9) *with $(\tilde q, \tilde v, \tilde\vartheta) \in \dot\Lambda_\epsilon^{4,5,5}$ and satisfies* (2.13)*, then (q, v, ϑ) satisfies the estimate*:

$$\sum_{\nu=0}^{1}\big\|(1+|x|)^2\nabla^\nu q\big\|_{L_\infty}+\sum_{\nu=0}^{1}\big\|(1+|x|)^{\nu+1}\nabla^\nu(v,\vartheta)\big\|_{L_\infty}$$
$$+\big\|(1+|x|)^2\nabla^2(v,\vartheta)\big\|_{L_\infty}\leqslant C\big\{\epsilon^2+\bar K\big\},\tag{2.17}$$

where the constant $C>0$ depends only on $\bar\rho$, $\bar\theta$, μ, μ' and κ.

Proof. First, we need to deduce an estimate on h. Since $(\tilde q,\tilde v,\tilde\vartheta)\in\dot\Lambda_\epsilon^{4,5,5}$, there exist $\tilde V_1=(\tilde V_1^i)_{1\leqslant i\leqslant 3}$ and $\tilde V_2$ such that $\nabla\cdot\tilde v=\nabla\cdot\tilde V_1+\tilde V_2$, and

$$\big\|(1+|x|)^3\tilde V_1\big\|_{L_\infty}+\big\|(1+|x|)^{-1}\tilde V_2\big\|_{L_1}\leqslant\epsilon\tag{2.18}$$

and thus we have

$$\begin{aligned}f^i&=-\tilde\rho(\tilde v\cdot\nabla)\tilde v^i+\tilde\rho F^i\\&=\nabla\cdot\big(-\tilde\rho\tilde v^i\tilde v+\tilde\rho\tilde v^i\tilde V_1+\tilde\rho F_1^i\big)+\big\{-\tilde\rho(\tilde V_1\cdot\nabla)\tilde v^i-\tilde V_1\cdot\nabla\tilde\rho\tilde v^i+\tilde\rho\tilde v^i\tilde V_2-\nabla\tilde\rho\cdot F_1^i+\tilde\rho F_2^i\big\}\\&=\nabla\cdot f_1^i+f_2^i.\end{aligned}$$

Due to the fact that $(\tilde q,\tilde v,\tilde\vartheta)\in\dot\Lambda_\epsilon^{4,5,5}$, (2.13) and by using the Sobolev inequality and mean value theorem, we have

$$\big\|(1+|x|)^3f^i\big\|_{L_\infty}+\big\|(1+|x|)^2f_1^i\big\|_{L_\infty}+\big\|f_2^i\big\|_{L_1}\leqslant C\big\{\epsilon^2+K_1\big\}.$$

Moreover, direct calculations show that

$$\big\|(1+|x|)^3\nabla f^i\big\|_{L_\infty}+\big\|(1+|x|)^2f^i\big\|_{L_\infty}\leqslant C\big\{\epsilon^2+K_1\big\},$$

where K_1 is defined by

$$K_1=\big\|(1+|x|)^3(F,\nabla F)\big\|+\big\|(1+|x|)^2F_1\big\|_{L_\infty}+\|F_2\|_{L_1}.$$

Hence, by (i) and (ii) of Lemma 2.4, we obtain

$$|x|\big|\omega^j\big|,\,|x|^2\big(|\Phi|,|\nabla\Phi|,\big|\nabla\omega^j\big|,\big|\nabla^2\omega^j\big|\big)\leqslant C\big\{\epsilon^2+K_1\big\}.\tag{2.19}$$

As for ∇p, ∇^2p and ∇^3p, direct calculations lead to

$$\begin{aligned}R&=-\sum_{i=1}^{3}\frac{\tilde\rho_p}{\tilde\rho}\tilde v_i\nabla_iq+\left\{-\sum_{i=1}^{3}\frac{\tilde\rho_\theta}{\tilde\rho}\tilde v_i\nabla_i\tilde\vartheta+\frac{G}{\tilde\rho}\right\}\\&\equiv-\sum_{i=1}^{3}q_{11}^iq_{12}^i+r_1,\end{aligned}$$

$$
\begin{aligned}
\nabla R = &-\sum_{i=1}^{3}\frac{\tilde\rho_p}{\tilde\rho}\tilde v^i\nabla_i(\nabla q)+\Bigg\{-\sum_{i=1}^{3}\bigg(\frac{\tilde\rho_p}{\tilde\rho}\nabla\tilde v^i\nabla_i q+\frac{\tilde\rho_\theta}{\tilde\rho}\nabla\tilde v^i\nabla_i\tilde\vartheta\bigg)\\
&+\nabla\bigg(\frac{\tilde\rho_p}{\tilde\rho}\bigg)(\tilde v\cdot\nabla)q+\nabla\bigg(\frac{\tilde\rho_\theta}{\tilde\rho}\bigg)(\tilde v\cdot\nabla)\tilde\vartheta+\frac{\tilde\rho_\theta}{\tilde\rho}(\tilde v\cdot\nabla)(\nabla\tilde\vartheta)+\frac{\nabla\tilde\rho}{\tilde\rho^2}G+\frac{1}{\tilde\rho}\nabla G\Bigg\}\\
\equiv &-\sum_{i=1}^{3}q_{21}^i q_{22}^i+r_2.
\end{aligned}
$$

Since $(\tilde q,\tilde v,\tilde\vartheta)\in\dot\Lambda_\epsilon^{4,5,5}$ and (2.13) holds, by using the Sobolev inequality, we can deduce that

$$
\begin{aligned}
&\big\|(1+|x|)^2 q_{j1}^i q_{j2}^i\big\|_{L_\infty}+\big\|(1+|x|)^3(\nabla q_{j1}^i)q_{j2}^i\big\|_{L_\infty}+\big\|(1+|x|)^3 q_{j1}^i(\nabla q_{j2}^i)\big\|_1\\
&\quad\leqslant C\big\{\epsilon^2+K_0+K_2\big\},\\
&\big\|(1+|x|)^3\nabla r_j\big\|_{L_\infty}+\big\|(1+|x|)^2 r_j\big\|_{L_\infty}\leqslant C\big\{\epsilon^2+K_0+K_2\big\},
\end{aligned}
$$

for $j=1,2$. Here K_2 is defined by

$$
K_2=\big\|(1+|x|)^2 G\big\|_{L_\infty}+\big\|(1+|x|)^3\nabla G\big\|_{L_\infty}+\big\|(1+|x|)^3\nabla^2 G\big\|_{L_\infty}.
$$

This implies

$$
|x|\big|\nabla p(x)\big|,\ |x|^2\big(\big|\nabla^2 p(x)\big|,\big|\nabla^3 p(x)\big|\big)\leqslant C\big\{\epsilon^2+K_0+K_2\big\}, \tag{2.20}
$$

as follows from (ii) of Lemma 2.4.

Combining (2.14), (2.19) and (2.20), we deduce that

$$
|x||v|,\ |x|^2\big(|\nabla v|,\big|\nabla^2 v\big|,|q|,|\nabla q|\big)\leqslant C\big\{\epsilon^2+K_0+K_1+K_2\big\}. \tag{2.21}
$$

Now we turn to deal with ϑ. In this case, we rewrite Θ as

$$
\begin{aligned}
\Theta&=\frac{1}{\bar\kappa}\big\{-\eta_1(\tilde v\cdot\nabla)\tilde\vartheta-\eta_2(\tilde v\cdot\nabla)\tilde q+\Psi(\tilde v)+\eta_3 G+H\big\}\\
&=\frac{1}{\bar\kappa}\nabla\cdot\big\{-(\eta_1\tilde\vartheta+\eta_2\tilde q)\tilde v+(\eta_1\tilde\vartheta+\eta_2\tilde q)\tilde V_1+\eta_3 G_1+H_1\big\}+\frac{1}{\bar\kappa}\big\{-(\tilde V_1\cdot\nabla)(\eta_1\tilde\vartheta+\eta_2\tilde q)\\
&\quad+(\eta_1\tilde\vartheta+\eta_2\tilde q)\cdot\tilde V_2-\tilde\vartheta\tilde v\cdot\nabla\eta_1+\tilde q\tilde v\cdot\nabla\eta_2+\Psi(\tilde v)-\nabla\eta_3\cdot G_1+\eta_3 G_2+H_2\big\}\\
&\equiv\nabla\cdot\Theta_1+\Theta_2,
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta&=\sum_{i=1}^{3}\big(-\eta_2\tilde v^i\nabla_i\tilde q\big)+\frac{1}{\bar\kappa}\big\{-\eta_1(\tilde v\cdot\nabla)\tilde\vartheta+\Psi(\tilde v)+\eta_3 G\big\}+H\\
&\equiv\sum_{i=1}^{3}\Theta_1^i\Theta_2^i+\Theta_3,
\end{aligned}
$$

also by $(\tilde{q}, \tilde{v}, \tilde{\vartheta}) \in \dot{\Lambda}_{\epsilon}^{4,5,5}$ and (2.13), it follows from (2.13) and the Sobolev inequality that

$$\begin{aligned}
&\left\|\left(1+|x|\right)^{3}\Theta\right\|_{L_\infty} + \left\|\left(1+|x|\right)^{2}\Theta_1\right\|_{L_\infty} + \|\Theta_2\|_{L_1} \leqslant C\left\{\epsilon^2 + K_3\right\},\\
&\left\|\left(1+|x|\right)^{2}\Theta_1^i\Theta_2^i\right\|_{L_\infty} + \left\|\left(1+|x|\right)^{3}\left(\nabla\Theta_1^i\right)\Theta_2^i\right\|_{L_\infty} + \left\|\left(1+|x|\right)^{3}\Theta_1^i\left(\nabla\Theta_2^i\right)\right\|_{1} \leqslant C\left\{\epsilon^2 + K_3\right\},\\
&\left\|\left(1+|x|\right)^{3}\nabla\Theta_3\right\|_{L_\infty} + \left\|\left(1+|x|\right)^{2}\Theta_3\right\|_{L_\infty} \leqslant C\left\{\epsilon^2 + K_3\right\},
\end{aligned}$$

where K_3 is defined by

$$K_3 = \left\|\left(1+|x|\right)^{3}(G, H, \nabla G, \nabla H)\right\|_{L_\infty} + \left\|\left(1+|x|\right)^{2}(G_1, H_1)\right\|_{L_\infty} + \left\|(G_2, H_2)\right\|_{L_1}.$$

This implies

$$|x||\vartheta|, |x|^2\left(|\nabla\vartheta|, \left|\nabla^2\vartheta\right|\right) \leqslant C\left\{\epsilon^2 + K_3\right\} \tag{2.22}$$

as follows from (i) and (ii) of Lemma 2.4.

Next, we consider the case where $|x| < 1$. The Sobolev inequality and the Hardy inequality imply that

$$\left(|q|, |v|, |\vartheta|\right) \leqslant 2\left\|\frac{(q, v, \vartheta)}{1+|x|}\right\|_{L_\infty} \leqslant C\left\|\frac{(q, v, \vartheta)}{1+|x|}\right\|_{2} \leqslant C\left\|(\nabla q, \nabla v, \nabla\vartheta)\right\|_{1,1,1} \leqslant C\left\{\epsilon^2 + K_0\right\}. \tag{2.23}$$

While for $\nu = 1, 2$

$$\left|\nabla^\nu(q, v, \vartheta)\right| \leqslant C\left\|\nabla^\nu(q, v, \vartheta)\right\|_2 \leqslant C\left\{\epsilon^2 + K_0\right\}. \tag{2.24}$$

Having obtained (2.21)–(2.24), (2.17) follows immediately. This completes the proof of Lemma 2.5. □

The main purpose of the following proposition is to prove that $(q, v, \vartheta) \in \dot{\Lambda}_{\epsilon}^{4,5,5}$.

Proposition 2.1. *There exists $c_0 > 0$ such that for any sufficiently small $\epsilon > 0$, if $(G, F, H) \in \mathcal{H}^{4,3,3}$ satisfies*

$$K \equiv \bar{K} + \left\|\left(1+|x|\right)^{-1}G\right\|_{L_1} \leqslant c_0\epsilon \quad (\bar{K} \text{ is defined in Lemma } 2.5), \tag{2.25}$$

then (2.9) *with $(\tilde{q}, \tilde{v}, \tilde{\vartheta}) \in \dot{\Lambda}_{\epsilon}^{4,5,5}$ admits a solution $(q, v, \vartheta) = T(\tilde{q}, \tilde{v}, \tilde{\vartheta}) \in \dot{\Lambda}_{\epsilon}^{4,5,5}$.*

Proof. By Lemmas 2.3 and 2.5, it follows that (2.9) has a solution $(q, v, \vartheta) \in \hat{\mathcal{H}}^{4,5,5}$, which satisfies

$$\left\|(q, v, \theta)\right\|_{\Lambda^{4,5,5}} \leqslant C\epsilon^2 + \bar{K} \leqslant C\epsilon^2 + c_0\epsilon,$$

where the constant $C>0$ depends only on ρ,θ,μ,μ' and κ. Thus if we take $c_0\leqslant\frac{1}{2C}$ and $\epsilon>0$ is small enough, it follows that $(q,v,\vartheta)\in\Lambda_\epsilon^{4,5,5}$. And then, we define V_1 and V_2 by

$$V_1=-\frac{\tilde\rho_P}{\tilde\rho}\tilde v q,\qquad V_2=\nabla\cdot\left(\frac{\tilde\rho_P}{\tilde\rho}\tilde v\right)q-\frac{\tilde\rho_\theta}{\tilde\rho}\tilde v\cdot\nabla\tilde\vartheta+\frac{G}{\tilde\rho},$$

then by $(2.9)_1$ we can get

$$\nabla\cdot v=\nabla\cdot V_1+V_2.$$

Moreover, by $(\tilde q,\tilde v,\tilde\vartheta)\in\dot\Lambda_\epsilon^{4,5,5}$ and (2.13), we have from Sobolev's inequality that

$$\begin{aligned}\big\|\big(1+|x|\big)^3V_1\big\|_{L_\infty}+\big\|\big(1+|x|\big)^{-1}V_2\big\|_{L_1}&\leqslant C\big\{\epsilon^2+\bar K+\big\|\big(1+|x|\big)^{-1}G\big\|_{L_1}\big\}\\&\leqslant C\big\{\epsilon^2+c_0\epsilon\big\}\leqslant C\epsilon^2+\frac{\epsilon}{2}\leqslant\epsilon,\end{aligned}$$

if $c_0\leqslant\frac{1}{2C}$ and $\epsilon>0$ are taken sufficiently small. This completes the proof of Proposition 2.1. □

2.2.2. Contraction of the solution map T

In this subsection, we shall show that the solution map T for (2.9) is contractive. Suppose that $(\tilde q^j,\tilde v^j,\tilde\vartheta^j)\in\dot\Lambda_\epsilon^{4,5,5}$ and $(q^j,v^j,\vartheta^j)=T(\tilde q^j,\tilde v^j,\tilde\vartheta^j)$ for $j=1,2$, it is easy to deduce from (2.9) that

$$\begin{cases}\nabla\cdot\big(v^1-v^2\big)+\left(\dfrac{\tilde\rho_P^1}{\tilde\rho^1}\tilde v^1\cdot\nabla\right)\big(q^1-q^2\big)=g,\\-\mu\Delta\big(v^1-v^2\big)-(\mu+\mu')\nabla\big(\nabla\cdot\big(v^1-v^2\big)\big)+\nabla\big(q^1-q^2\big)=\tilde f,\\-\kappa\Delta\big(\vartheta^1-\vartheta^2\big)=-\eta_1^1\big(\tilde v^1\cdot\nabla\big)\big(\tilde\vartheta^1-\tilde\vartheta^2\big)+\tilde h,\end{cases}\tag{2.26}$$

where $(g,h,f)\in\mathcal H^{3,3,3}$ is defined by

$$\begin{cases}g=-\left(\dfrac{\tilde\rho_\theta^1}{\tilde\rho^1}\tilde v^1-\dfrac{\tilde\rho_\theta^2}{\tilde\rho^2}\tilde v^2\right)\cdot\nabla q^2-\left(\dfrac{\tilde\rho_\theta^1}{\tilde\rho^1}\tilde v^1\cdot\nabla\tilde\vartheta^1-\dfrac{\tilde\rho_\theta^2}{\tilde\rho^2}\tilde v^2\cdot\nabla\tilde\vartheta^2\right)+\left(\dfrac{1}{\tilde\rho^1}-\dfrac{1}{\tilde\rho^2}\right)G,\\\tilde f=-\big(\tilde\rho^1\big(\tilde v^1\cdot\nabla\big)\tilde v^1-\tilde\rho^2\big(\tilde v^2\cdot\nabla\big)\tilde v^2\big)+\big(\tilde\rho^1-\tilde\rho^2\big)F,\\\tilde h=-\big(\eta_1^1v^1-\eta_1^2v^2\big)\cdot\nabla\tilde\vartheta^2-\big(\eta_2^1\big(\tilde v^1\cdot\nabla\big)\tilde q^1-\eta_2^2\big(\tilde v^2\cdot\nabla\big)\tilde q^2\big)+\big(\eta_3^1-\eta_3^2\big)G,\\\eta_1^j=\tilde\rho^j\tilde c_\nabla^j+\dfrac{\tilde\theta^j\tilde\rho_\theta^{j2}}{\tilde\rho^j\tilde\rho_P^j},\quad\eta_2^j=\dfrac{\tilde\theta^j\tilde\rho_\theta^j}{\tilde\rho^j},\quad\eta_3^j=\dfrac{\tilde\theta^j\tilde\rho_\theta^j}{\tilde\rho^j\tilde\rho_P^j},\end{cases}\tag{2.27}$$

for $j=1,2$. Since

$$\begin{aligned}&\big\|\big(1+|x|\big)g\big\|+\sum_{\nu=1}^{3}\big\|\big(1+|x|\big)^\nu\nabla^\nu g\big\|+\sum_{\nu=0}^{2}\big\|\big(1+|x|\big)^{\nu+1}\nabla^\nu(h,f)\big\|\\&\quad\leqslant C\{\epsilon+K_0\}\big\|\big(\tilde q^1-\tilde q^2,\tilde v^1-\tilde v^2,\tilde\vartheta^1-\tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}},\end{aligned}\tag{2.28}$$

which follows from the Sobolev inequality with K_0 defined in (2.12). Setting

$$a = \frac{\tilde\rho_P^1}{\tilde\rho^1}\tilde v^1; \qquad b_1 = c_1 = 0; \qquad b_2 = \eta_1^1 \tilde v^1, \qquad c_2 = \tilde\vartheta^1 - \tilde\vartheta^2,$$

and g, $\tilde f$, $\tilde h$ as defined in (2.27), we have by applying Theorem 2.1 with $k = 3$ to (2.26) that

$$\begin{aligned}
&\big\|\big(q^1 - q^2, v^1 - v^2, \vartheta^1 - \vartheta^2\big)\big\|_{L_6} \\
&\quad + \sum_{\nu=1}^{3}\big\|\big(1+|x|\big)^{\nu}\nabla^{\nu}\big(q^1 - q^2\big)\big\| + \sum_{\nu=1}^{4}\big\|\big(1+|x|\big)^{\nu-1}\big(\nabla^{\nu}\big(v^1 - v^2\big), \nabla^{\nu}\big(\vartheta^1 - \vartheta^2\big)\big)\big\| \\
&\leqslant C\{\epsilon + K_0\}\big\|\big(\tilde q^1 - \tilde q^2, \tilde v^1 - \tilde v^2, \tilde\vartheta^1 - \tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}}.
\end{aligned} \tag{2.29}$$

Similarly, by using the same argument as in the proof of Lemma 2.5, we can get

$$\begin{aligned}
&\big\|\big(1+|x|\big)^2\big(q^1 - q^2\big)\big\|_{L_\infty} \\
&\quad + \sum_{\nu=0}^{1}\big\|\big(1+|x|\big)^{\nu+1}\nabla^{\nu}\big(v^1 - v^2\big)\big\|_{L_\infty} + \sum_{\nu=1}^{2}\big\|\big(1+|x|\big)^{\nu}\nabla^{\nu}\big(\vartheta^1 - \vartheta^2\big)\big\|_{L_\infty} \\
&\leqslant C\{\epsilon + \bar K\}\big\|\big(\tilde q^1 - \tilde q^2, \tilde v^1 - \tilde v^2, \tilde\vartheta^1 - \tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}} \\
&\quad + C\epsilon\big\{\big\|\big(1+|x|\big)^3\big(\tilde V_1^1 - \tilde V_2^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(\tilde V_1^1 - \tilde V_2^2\big)\big\|_{L_1}\big\},
\end{aligned} \tag{2.30}$$

where $\tilde V_1^j$, $\tilde V_2^j$ $(j = 1, 2)$ are functions satisfying

$$\nabla\cdot\tilde v^j = \nabla\cdot\tilde V_1^j + \tilde V_2^j, \qquad \big\|\big(1+|x|\big)^3\tilde V_1^j\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\tilde V_2^j\big\|_{L_1} \leqslant \epsilon. \tag{2.31}$$

Moreover, if we define V_1^j, V_2^j $(j = 1, 2)$ as

$$V_1^j = -\frac{\tilde\rho_P^j}{\tilde\rho^j}\tilde v^j q^j, \qquad V_2^j = \nabla\cdot\left(\frac{\tilde\rho_P^j}{\tilde\rho^j}\tilde v^j\right)q^j - \frac{\tilde\rho_\theta^j}{\tilde\rho^j}\tilde v^j\cdot\nabla\tilde\vartheta^j + \frac{G}{\tilde\rho^j}, \tag{2.32}$$

then

$$\begin{aligned}
&\big\|\big(1+|x|\big)^3\big(V_1^1 - V_1^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(V_2^1 - V_2^2\big)\big\|_{L_1} \\
&\quad \leqslant C\big\{\epsilon + \big\|\big(1+|x|\big)^{-1}G\big\|_{L_1}\big\}\big\|\big(\tilde q^1 - \tilde q^2, \tilde v^1 - \tilde v^2, \tilde\vartheta^1 - \tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}}.
\end{aligned} \tag{2.33}$$

As for $\|(\vartheta^1 - \vartheta^2)\|_{L_\infty}$, it can be estimated by the following inequality:

$$\big\|\big(\vartheta^1 - \vartheta^2\big)\big\|_{L_\infty} \leqslant C\sum_{\nu=0}^{1}\big\|\nabla^{\nu}\big(\vartheta^1 - \vartheta^2\big)\big\|_{L_6} \leqslant C\big\|\nabla\big(\vartheta^1 - \vartheta^2\big)\big\|_1, \tag{2.34}$$

which follows from the Sobolev inequality. Combining (2.29)–(2.34) we can obtain

$$\begin{aligned}
&\big\|\big(q^1-q^2, v^1-v^2, \vartheta^1-\vartheta^2\big)\big\|_{\Lambda^{3,4,4}} \\
&\quad + \big\|\big(1+|x|\big)^3\big(V_1^1-V_1^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(V_2^1-V_2^2\big)\big\|_{L_1} \\
&\quad \leqslant C\{\epsilon+K\}\big\|\big(\tilde q^1-\tilde q^2, \tilde v^1-\tilde v^2, \tilde\vartheta^1-\tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}} \\
&\quad + C\epsilon\big\{\big\|\big(1+|x|\big)^3\big(\tilde V_1^1-\tilde V_1^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(\tilde V_2^1-\tilde V_2^2\big)\big\|_{L_1}\big\}. \qquad (2.35)
\end{aligned}$$

Therefore, we have the following proposition.

Proposition 2.2. *There exists $c_0>0$ such that for any ϵ sufficiently small, if $(G,F,H)\in\mathcal{H}^{4,3,3}$ satisfies*

$$K \leqslant c_0\epsilon \quad (K \text{ is defined in Proposition } 2.1),$$

then for $(\tilde q^j,\tilde v^j,\tilde\vartheta^j)\in\dot\Lambda_\epsilon^{4,5,5}$ and $(q^j,v^j,\vartheta^j)=T(\tilde q^j,\tilde v^j,\tilde\vartheta^j)$ $(j=1,2)$, we have the following estimate:

$$\begin{aligned}
&\big\|\big(q^1-q^2, v^1-v^2, \vartheta^1-\vartheta^2\big)\big\|_{\Lambda^{3,4,4}} \\
&\quad + \big\|\big(1+|x|\big)^3\big(V_1^1-V_1^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(V_2^1-V_2^2\big)\big\|_{L_1} \\
&\quad \leqslant \frac{1}{2}\big\{\big\|\big(\tilde q^1-\tilde q^2, \tilde v^1-\tilde v^2, \tilde\vartheta^1-\tilde\vartheta^2\big)\big\|_{\Lambda^{3,4,4}} \\
&\quad + \big\|\big(1+|x|\big)^3\big(\tilde V_1^1-\tilde V_1^2\big)\big\|_{L_\infty} + \big\|\big(1+|x|\big)^{-1}\big(\tilde V_2^1-\tilde V_2^2\big)\big\|_{L_1}\big\}, \qquad (2.36)
\end{aligned}$$

where $(\tilde V_1^j,\tilde V_2^j)$ $(j=1,2)$ satisfy (2.31)*, and V_1^j, V_2^j are defined by* (2.32).

Hence, by Propositions 2.1 and 2.2, the contraction mapping principle implies the existence and uniqueness of solution to (1.3). This completes the proof of Theorem 1.1.

3. Non-stationary problem

In this part, we consider the stability of the stationary solution with respect to the initial disturbance $(\rho_0, v_0, \vartheta_0)$. Fix $\bar\rho$, $\bar\theta$ to be positive constants and let F, G be small in the sense of Theorem 1.1. We denote the corresponding stationary solution obtained in Theorem 1.1 by (P^*, v^*, θ^*), and set $\rho^*\equiv\bar\rho+\sigma^*=\rho(P^*,\theta^*)$. Then by direct calculations we have the following estimate for σ^*:

$$\begin{aligned}
\|\sigma^*\|_{\mathcal{J}^4} &\equiv \sum_{\nu=1}^{4}\big\|\big(1+|x|\big)^{\nu-1}\nabla^\nu\sigma^*\big\| + \sum_{\nu=0}^{1}\big\|\big(1+|x|\big)^{\nu+1}\nabla^\nu\sigma^*\big\|_{L_\infty} + \big\|\big(1+|x|\big)^2\nabla^2\sigma^*\big\|_{L_\infty} \\
&\leqslant C\epsilon,
\end{aligned}$$

where C is depending only on $\bar\rho$ and $\bar\theta$. So we have

$$\|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \equiv \|\sigma^*\|_{\ddot{J}^4} + \|v^*\|_{J^5} + \|\vartheta^*\|_{J^5} \leqslant (C+1)\epsilon.$$

Thus for simplicity, we can assume in this part $\|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$ for ϵ small enough. By putting

$$\rho(t,x) = \rho^* + \sigma(t,x), \qquad v(t,x) = v^* + w(t,x), \qquad \theta(t,x) = \theta^* + \vartheta(t,x)$$

into (1.1), we have the following system of equations for (σ, w, ϑ):

$$\begin{cases} \sigma_t(t) + \nabla\cdot\{(\rho^*+\sigma(t))w(t)\} = -\nabla\cdot(v^*\sigma(t)), \\ w_t(t) - \dfrac{1}{\rho^*}\big[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\big] + A(t)\nabla\sigma(t) + B(t)\nabla\theta(t) = f(t), \\ \vartheta_t(t) - \kappa E^*\Delta\vartheta(t) + D(t)\nabla\cdot w(t) = h(t), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f(t) &= -(v^*\cdot\nabla)w(t) - (w(t)\cdot\nabla)(v^*+w(t)) - (A(t)-A^*)\nabla\rho^* - (B(t)-B^*)\nabla\theta^* \\ &\quad - \frac{\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\big[\mu\Delta(v^*+w(t)) + (\mu+\mu')\nabla\{\nabla\cdot(v^*+w(t))\}\big], \\ h(t) &= -(v^*\cdot\nabla)\theta(t) - (w(t)\cdot\nabla)(\theta^*+\theta(t)) + \kappa(E(t)-E^*)\Delta(\theta^*+\theta(t)) \\ &\quad - (D(t)-D^*)\nabla\cdot v^* + (E(t)-E^*)\Psi(v^*) \\ &\quad + E(t)(\Psi(v^*+w(t)) - \Psi(v^*)) + (E(t)-E^*)H, \end{aligned}$$

and

$$A = \frac{P_\rho(\rho,\theta)}{\rho}, \qquad B = \frac{P_\theta(\rho,\theta)}{\rho}, \qquad D = \frac{\theta P_\theta(\rho,\theta)}{\rho c_\nabla(\rho,\theta)}, \qquad E = \frac{1}{\rho c_\nabla(\rho,\theta)},$$

$A^* = A(\rho^*,\theta^*)$, $E(t) = E(\rho^*+\sigma(t), \theta^*+\vartheta(t))$, etc. Moreover, we set $A_i(t)$, $B_i(t)$, $D_i(t)$, $E_i(t)$, $i=1,2$, to be the functions satisfying: $A(t)-A^* = A_1(t)\sigma(t) + A_2(t)\vartheta(t)$, $\ldots$, $E(t)-E^* = E_1(t)\sigma(t)+E_2(t)\vartheta(t)$.

In this section, we will prove Theorem 1.2. The proof consists of the following two steps. The first one is the local existence result:

Proposition 3.1. *If* $(\sigma, w, \vartheta)(0) \in \mathcal{H}^{3,3,3}$, *then there exists* $t_0 > 0$ *such that the initial value problem* (3.2) *with initial data* $(\sigma, w, \vartheta)(0)$ *admits a unique solution* $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_0; \mathcal{H}^{3,3,3})$. *Moreover,* $(\sigma, w, \vartheta)(t)$ *satisfies*

$$\|(\sigma, w, \vartheta)(t)\|_{3,3,3}^2 \leqslant 2\|(\sigma, w, \vartheta)(0)\|_{3,3,3}^2$$

for any $t \in [0, t_0]$.

And the other is an a priori estimate:

Proposition 3.2. *Let $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{3,3,3})$ be a solution to* (3.1). *Then there exists $\epsilon_0 > 0$ such that if $\epsilon \leqslant \epsilon_0$ and $\sup_{0\leqslant t\leqslant t_1} \|(\sigma, w, \vartheta)(t)\|_{3,3,3}, \|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$, then*

$$\big\|(\sigma, w, \vartheta)(t)\big\|_{3,3,3}^2 + \int_0^t \big\|(\nabla\sigma, \nabla w, \nabla\vartheta)(s)\big\|_{2,3,3}^2 + \big\|(w_t, \theta_t)(s)\big\|_{2,2}^2\, ds \leqslant C\big\|(\sigma, w, \vartheta)(0)\big\|_{3,3,3}^2 \tag{3.2}$$

for any $t \in [0, t_1]$, where $C > 0$ is a constant depending only on μ, μ' and κ.

Concerning the local existence result, we can apply the Matsumura and Nishida [13] method directly. So we shall devote the following sections to the proof of Proposition 3.2.

Before starting to estimate the solution, let us introduce the absolute constant $\bar{\epsilon} > 0$ such that $C\bar{\epsilon} = 1/4 \min\{\bar{\rho}, \bar{\theta}\}$, where C is the constant which appears in the embedding: $\|\cdot\|_{L_\infty} \leqslant C\|\cdot\|_2$. Moreover, we denote the domain of (ρ, θ) by $S(\bar{\rho}, \bar{\theta})$: $S(\bar{\rho}, \bar{\theta}) = \{(\rho, \theta) \mid \bar{\rho}/2 \leqslant \rho \leqslant 3/2\bar{\rho},\ \bar{\theta}/2 \leqslant \theta \leqslant 3/2\bar{\theta}\}$.

3.1. Some estimates for $f(t)$, $h(t)$ and their derivatives

This subsection is devoted to deducing some estimates for $f(t)$, $h(t)$ and their derivatives which are collected in the following lemma.

Lemma 3.1. *Let $(\sigma, w, \vartheta)(t)$ and $(\sigma^*, v^*, \vartheta^*)$ be satisfying*

$$\big\|(\sigma, w, \vartheta)(t)\big\|_{3,3,3}, \big\|(\sigma^*, v^*, \vartheta^*)\big\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon \leqslant \bar{\epsilon}.$$

Then for a multi-index α with $0 \leqslant |\alpha| \leqslant 3$, if we write $\partial_x^\alpha h(t)$ of the form:

$$\partial_x^\alpha f(t) = -\frac{\sigma(t)}{\rho^*(\rho^* + \sigma(t))}\big[\mu\Delta\partial_x^\alpha w(t) + (\mu + \mu')\nabla\big(\nabla \cdot \partial_x^\alpha w(t)\big)\big] + F_\alpha(t),$$

then we have the estimate for $H_\alpha(t)$:

$$F_\alpha(t) \leqslant C \begin{cases} |\nabla v^*||w(t)| + (|v^*| + |w(t)|)|\nabla w(t)| + (|\nabla\rho^*| + |\nabla\theta^*| + |\nabla^2 v^*|)|\sigma(t)| \\ \quad + (|\nabla\rho^*| + |\nabla\sigma^*|)|\vartheta(t)|, \quad \text{if } \alpha = 0, \\ |\nabla^{|\alpha|+1} v^*||w(t)| + \sum_{\nu=1}^{|\alpha|+1} |\nabla^\nu w(t)| + \sum_{\nu=1}^{|\alpha|+1} (|\nabla^\nu\rho^*| + |\nabla^\nu\theta^*| \\ \quad + |\nabla^{\nu+1} v^*|)|\sigma(t)| + \sum_{\nu=1}^{|\alpha|} |\nabla^\nu\sigma(t)| + \sum_{\nu=1}^{|\alpha|+1} (|\nabla^\nu\rho^*| + |\nabla^\nu\theta^*|)|\vartheta(t)| \\ \quad + \sum_{\nu=1}^{|\alpha|} |\nabla^\nu\vartheta(t)| + R_F^{|\alpha|}(t), \quad \text{if } |\alpha| = 1, 2, 3. \end{cases} \tag{3.3}$$

Here $R_F^1(t) = 0$ and $R_F^k(t)$ $(k = 2, 3)$ satisfies:

$$\big\|R_F^2(t)\big\|_{L_{3/2}} \leqslant C\epsilon\big\|\nabla^2\sigma(t)\big\|, \qquad \big\|R_F^2(t)\big\| \leqslant C\epsilon\big\|\nabla^3\sigma(t)\big\|,$$

$$\big\|R_F^3(t)\big\|_{L_{3/2}} \leqslant C\epsilon\big\|\big(\nabla^2\sigma, \nabla^3 w, \nabla^2\vartheta\big)(t)\big\|_{1,0,0}.$$

And if writing $\partial_x^\alpha h$ of the form:

$$\partial_x^\alpha h = \kappa\big(E_1(t)\sigma(t) + E_2(t)\vartheta(t)\big)\Delta\big(\partial_x^\alpha \vartheta(t)\big) + H_\alpha(t),$$

we have the following estimate for $F_\alpha(t)$:

$$H_\alpha(t) \leqslant C \begin{cases} (|\nabla^2\theta^*| + |\nabla v^*| + |H|)|\sigma(t)| + (|\nabla^2\theta^*| + |\nabla v^*| + |H|)|\vartheta(t)| + |v^*||\nabla\vartheta(t)| \\ \quad + (|\nabla\theta^*| + |\nabla\theta(t)|)|w(t)| + (|\nabla w(t)| + |v^*|)|\nabla w(t)|, \quad \text{if } |\alpha| = 0, \\ \sum_{\nu=1}^{|\alpha|+1}(|\nabla^{\nu+1}\theta^*| + |\nabla^\nu v^*| + |\nabla^{\nu-1}H|)(|\vartheta(t)| + |\sigma(t)|) \\ \quad + \sum_{\nu=1}^{|\alpha|+1}|\nabla^\nu w(t)| + \sum_{\nu=1}^{|\alpha|+1}|\nabla^\nu\vartheta(t)| + \sum_{\nu=1}^{|\alpha|}|\nabla^\nu\sigma(t)| + R_H^{|\alpha|}(t), \\ \quad \text{if } |\alpha| = 1, 2, 3. \end{cases} \tag{3.4}$$

Here $R_H^1(t) = 0$ and $R_H^k(t)$ $(k = 2, 3)$ satisfies:

$$\big\|R_H^2(t)\big\|_{L_{3/2}} \leqslant C\epsilon\big\|\nabla^2 w(t)\big\|, \qquad \big\|R_H^2(t)\big\| \leqslant C\epsilon\big\|\nabla^3 w(t)\big\|,$$
$$\big\|R_H^3(t)\big\|_{L_{3/2}} \leqslant C\epsilon\big\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\big\|_{0,2,0},$$

where the constant $C > 0$ is depending only on μ, μ', κ and $\bar\epsilon$.

Proof. By the Leibniz rule and the Sobolev embedding $H^2 \hookrightarrow L_\infty$, we can easily check (3.3), (3.4) with

$$R_F^k(t) = \begin{cases} 0, & \text{if } k = 1, \\ |\nabla^2 w(t)||\nabla^2\sigma(t)|, & \text{if } k = 2, \\ |\nabla^2 w(t)||\nabla^3\sigma(t)| + (|\nabla^2 w(t)| + |\nabla^3 w(t)|)|\nabla^2\sigma(t)| + |\nabla^2 w(t)|^2 & \\ \quad + (|\nabla^4 v^*| + |\nabla^3\rho^*|)|\nabla\sigma(t)| + |\nabla^3\rho^*|(|\nabla^2 w(t)| + |\nabla\vartheta(t)|), & \text{if } k = 3; \end{cases} \tag{3.5}$$

$$R_H^k(t) = \begin{cases} 0, & \text{if } k = 1, \\ |\nabla^2 w(t)|^2, & \text{if } k = 2, \\ (|\nabla^2\sigma(t)| + |\nabla^2\sigma(t)|)|\nabla^2 w(t)| + (|\nabla^2 w(t)| + |\nabla^3 w(t)|)|\nabla^2 w(t)| & \\ \quad + (|\nabla^4\theta^*| + |\nabla^2 H|)(|\nabla\sigma(t)| + |\nabla\vartheta(t)|) + |\nabla^4 v^*||\nabla w(t)|, & \text{if } k = 3. \end{cases} \tag{3.6}$$

Using the Gagliard–Nirenberg inequality, we have

$$\begin{aligned} \big\|R_F^2(t)\big\|_{L_{3/2}} &\leqslant \big\|\nabla^2 w(t)\big\|_{L_6}\big\|\nabla^2\sigma(t)\big\| \leqslant C\epsilon\big\|\nabla^2\sigma(t)\big\|, \\ \big\|R_F^3(t)\big\|_{L_{3/2}} &\leqslant \big\|\nabla^2 w(t)\big\|_{L_6}\big\|\nabla^3\sigma(t)\big\| + \big\|\nabla^2 w(t)\big\|_{L_6}\big\|\nabla^2\sigma(t)\big\| + \big\|\nabla^3 w(t)\big\|\big\|\nabla^2\sigma(t)\big\|_{L_6} \\ &\quad + \big\|\nabla^2 w(t)\big\|\big\|\nabla^2 w(t)\big\|_{L_6} + \big(\big\|\nabla^4 v^*\big\| + \big\|\nabla^3\rho^*\big\|\big)\big\|\nabla\sigma(t)\big\|_{L_6} \\ &\quad + \big\|\nabla^3\rho^*\big\|\big(\big\|\nabla^2 w(t)\big\|_{L_6} + \big\|\nabla\vartheta(t)\big\|_{L_6}\big) \\ &\leqslant C\epsilon\big\|\big(\nabla^2\sigma, \nabla^3 w, \nabla^2\vartheta\big)(t)\big\|_{1,0,0}, \end{aligned}$$

and by the Sobolev inequality,

$$\left\| R_F^2(t) \right\| \leqslant \left\| \nabla^2 w(t) \right\|_{L_3} \left\| \nabla^2 \sigma(t) \right\|_{L_6} \leqslant C \left\| \nabla^2 w(t) \right\|_1 \left\| \nabla^2 \sigma(t) \right\|_{L_6} \leqslant C\epsilon \left\| \nabla^3 \sigma(t) \right\|.$$

From (3.6), similarly we can also get the estimates for $R_H^k(t)$ ($k = 1, 2, 3$). This completes the proof of Lemma 3.1. $\square$

3.2. *Estimates for $\nabla w(t)$, $\nabla\vartheta(t)$ and their derivatives up to $\nabla^4 w(t)$, $\nabla^4\vartheta(t)$*

The purpose of this subsection is to estimate $\nabla w(t)$, $\nabla\vartheta(t)$ and their derivatives up to $\nabla^4 w(t)$, $\nabla^4\vartheta(t)$. For results in this direction, we have

Lemma 3.2. *Let $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{3,3,3})$ be a solution to* (3.1). *Then there exist $\epsilon_0 > 0$, $\lambda_0 > 0$ and $\alpha, \beta > 0$ such that if $\epsilon \leqslant \epsilon_0$ and $\|(\sigma, w, \vartheta)(t)\|_{3,3,3}, \|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$, the following estimates hold*:

$$\frac{d}{dt}\left[\left\|\sigma(t)\right\|^2 + \left(\tilde{A}(t)w(t), w(t)\right) + \left(\tilde{B}(t)\vartheta(t), \vartheta(t)\right)\right] + \alpha\left\|\nabla w(t)\right\|^2 + \beta\left\|\nabla\vartheta(t)\right\|^2 \leqslant C\epsilon\left\|\nabla\sigma(t)\right\|^2, \tag{3.7}$$

and for $1 \leqslant k \leqslant 3$ and any λ with $0 < \lambda < \lambda_0$

$$\begin{aligned}
&\frac{d}{dt}\left[\left\|\nabla^k\sigma(t)\right\|^2 + \left(\tilde{A}(t)\nabla^k w(t), \nabla^k w(t)\right) + \left(\tilde{B}(t)\nabla^k\vartheta(t), \nabla^k\vartheta(t)\right)\right] \\
&\quad + \alpha\left\|\nabla^{k+1} w(t)\right\|^2 + \beta\left\|\nabla^{k+1}\vartheta(t)\right\|^2 \\
&\leqslant C(\epsilon + \lambda)\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|_{k-1,k-1,k-1}^2 + C\lambda^{-1}\left\|\left(\nabla^k w, \nabla^k\vartheta\right)(t)\right\|^2,
\end{aligned} \tag{3.8}$$

where $C > 0$ is a constant depending only on μ, μ', and κ. Setting

$$\tilde{A} = \frac{\rho^2}{P_\rho(\rho, \theta)}, \qquad \tilde{B} = \frac{\rho^2 c_\nabla(\rho, \theta)}{\theta P_\rho(\rho, \theta)},$$

then $\tilde{A}(t) = \tilde{A}(\rho^* + \sigma(t), \theta^* + \vartheta(t))$ and $\tilde{B}(t) = \tilde{B}(\rho^* + \sigma(t), \theta^* + \vartheta(t))$.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{\infty,\infty,\infty})$. For any multi-index α with $0 \leqslant |\alpha| \leqslant 3$, applying ∂_x^α to $(3.1)_1$, $(3.1)_2$, $(3.1)_3$, multiplying the resultant equations by $\partial_x^\alpha\sigma(t)$, $\tilde{A}(t)\partial_x^\alpha w(t)$, $\tilde{B}(t)\partial_x^\alpha\vartheta(t)$, respectively, and integrating the final result with respect to x over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\partial_x^\alpha\sigma(t)\right\|^2 - \left(\left(\rho^* + \sigma(t)\right)\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\right) = \left(-\partial_x^\alpha\left(v^*\sigma(t)\right) - I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\right), \\
&\left(\tilde{A}(t)\partial_x^\alpha w_t(t), \partial_x^\alpha w(t)\right) - \left(\frac{\tilde{A}(t)}{\rho^*}\partial_x^\alpha\left\{\mu\Delta w(t) + (\mu + \mu')\nabla\left(\nabla \cdot w(t)\right)\right\}, \partial_x^\alpha w(t)\right) \\
&\quad + \left(\left(\rho^* + \sigma(t)\right)\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)\right) + \left(\tilde{A}(t)B(t)\nabla\partial_x^\alpha\vartheta(t), \partial_x^\alpha w(t)\right) \\
&\quad = \left(\partial_x^\alpha f(t) + J_\alpha^1(t), \tilde{A}(t)\partial_x^\alpha w(t)\right),
\end{aligned}$$

$$\begin{aligned}&\big(\tilde B(t)\partial_x^\alpha\vartheta_t(t),\partial_x^\alpha\vartheta(t)\big)-\big(\kappa E^*\tilde B(t)\partial_x^\alpha\vartheta(t),\Delta\partial_x^\alpha\vartheta(t)\big)+\big(\tilde A(t)B(t)\nabla\partial_x^\alpha w(t),\partial_x^\alpha\vartheta(t)\big)\\&\quad=\big(\partial_x^\alpha h(t)+J_\alpha^2(t),\tilde B(t)\partial_x^\alpha\vartheta(t)\big),\end{aligned}$$

where $I_\alpha(t)$ and $J_\alpha(t)$ are defined by

$$\begin{aligned}I_\alpha(t)&=\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_x^{\alpha-\beta}\big(\rho^*+\sigma(t)\big)\partial_x^\beta w(t),\\J_\alpha^1(t)&=\sum_{\beta<\alpha}\binom{\alpha}{\beta}\bigg[\bigg(\partial_x^{\alpha-\beta}\frac{1}{\rho^*}\bigg)\partial_x^\beta\big\{\mu\Delta w(t)+(\mu+\mu')\nabla\big(\nabla\cdot w(t)\big)\big\}\\&\qquad-\big(\partial_x^{\alpha-\beta}A(t)\big)\nabla\partial_x^\beta\sigma(t)-\big(\partial_x^{\alpha-\beta}B(t)\big)\nabla\partial_x^\beta\vartheta(t)\bigg],\\J_\alpha^2(t)&=\sum_{\beta<\alpha}\binom{\alpha}{\beta}\big[\kappa\big(\partial_x^{\alpha-\beta}E^*\big)\Delta\partial_x^\beta\vartheta(t)-\big(\partial_x^{\alpha-\beta}D(t)\big)\nabla\partial_x^\beta w(t)\big].\end{aligned}$$

Cancelling the terms $((\rho^*+\sigma(t))\nabla\partial_x^\alpha\sigma(t),\partial_x^\alpha w(t))$ and $(\tilde A(t)B(t)\nabla\partial_x^\alpha\vartheta(t),\partial_x^\alpha w(t))$ by adding the above three formulas, writing the first term of the second formula and the third formula as follows:

$$\begin{aligned}\big(\tilde A(t)\partial_x^\alpha w_t(t),\partial_x^\alpha w(t)\big)&=\frac12\frac{d}{dt}\big(\tilde A(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)-\frac12\big(\tilde A_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big),\\\big(\tilde B(t)\partial_x^\alpha\vartheta_t(t),\partial_x^\alpha\vartheta(t)\big)&=\frac12\frac{d}{dt}\big(\tilde B(t)\partial_x^\alpha\vartheta(t),\partial_x^\alpha\vartheta(t)\big)-\frac12\big(\tilde B_t(t)\partial_x^\alpha\vartheta(t),\partial_x^\alpha\vartheta(t)\big),\end{aligned}$$

and using integrations by parts, we can get

$$\begin{aligned}&\frac12\frac{d}{dt}\big\{\big\|\partial_x^\alpha\sigma(t)\big\|^2+\big(\tilde A(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)+\big(\tilde B(t)\partial_x^\alpha\vartheta(t),\partial_x^\alpha\vartheta(t)\big)\big\}\\&\qquad+\mu\bigg(\frac{\tilde A(t)}{\rho^*}\nabla\partial_x^\alpha w(t),\nabla\partial_x^\alpha w(t)\bigg)+\kappa\big(E^*\tilde B(t)\nabla\partial_x^\alpha\vartheta(t),\nabla\partial_x^\alpha\vartheta(t)\big)\\&\quad\leqslant\big|\big(\partial_x^\alpha\big(v^*\sigma(t)\big),\nabla\partial_x^\alpha\sigma(t)\big)\big|+\kappa\big|\big(\nabla\big(E^*\tilde B(t)\big)\cdot\nabla\partial_x^\alpha\vartheta(t),\partial_x^\alpha\vartheta(t)\big)\big|\\&\qquad+\big|\big(\partial_x^\alpha w(t),\nabla\big(\tilde A(t)B(t)\big)\partial_x^\alpha\vartheta(t)\big)\big|\\&\qquad+\bigg[\mu\bigg|\bigg(\nabla\partial_x^\alpha w(t),\nabla\bigg(\frac{\tilde A(t)}{\rho^*}\bigg)\partial_x^\alpha w(t)\bigg)\bigg|+(\mu+\mu')\bigg|\bigg(\partial_x^\alpha w(t)\cdot\nabla\bigg(\frac{\tilde A(t)}{\rho^*}\bigg),\nabla\cdot\partial_x^\alpha w(t)\bigg)\bigg|\bigg]\\&\qquad+\big|\big(\partial_x^\alpha f(t),\tilde A(t)\partial_x^\alpha w(t)\big)\big|+\big|\big(\partial_x^\alpha h(t),\tilde B(t)\partial_x^\alpha\vartheta(t)\big)\big|\\&\qquad+\big[\big|\big(I_\alpha(t),\nabla\partial_x^\alpha\sigma(t)\big)\big|+\big|\big(J_\alpha^1(t),\tilde A\partial_x^\alpha w(t)\big)\big|+\big|\big(J_\alpha^2(t),\tilde B\partial_x^\alpha\vartheta(t)\big)\big|\big]\\&\qquad+\frac12\big|\big(\tilde A_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)\big|+\frac12\big|\big(\tilde B_t(t)\partial_x^\alpha\vartheta(t),\partial_x^\alpha\vartheta(t)\big)\big|\\&\quad\equiv K_1+K_2+\cdots+K_9.\end{aligned}\tag{3.9}$$

Now, we estimate the right-hand side of (3.9) term by term by using the Sobolev inequality and the Gagliard–Nirenberg inequality. If $\alpha = 0$, employing the Hardy inequality, we have

$$K_1 \leqslant \big\|\big(1+|x|\big)v^*\big\|_{L_\infty} \left\|\frac{\sigma(t)}{|x|}\right\| \big\|\nabla\sigma(t)\big\| \leqslant C\epsilon\big\|\nabla\sigma(x)\big\|^2. \tag{3.10}$$

If $1 \leqslant |\alpha| \leqslant 3$, we can get by integrations by parts

$$\begin{aligned}
K_1 &\leqslant 2\big|\big(\nabla v^*\partial_x^\alpha\sigma(t), \partial_x^\alpha\sigma(t)\big)\big| + \sum_{\beta<\alpha}\big|\big(\nabla\partial_x^{\alpha-\beta}v^*\big(\partial_x^\beta\sigma(t)\big) - \big(\partial_x^{\alpha-\beta}v^*\big)\nabla\partial_x^\beta\sigma(t), \partial_x^\beta\sigma(t)\big)\big| \\
&\leqslant C\bigg\{\|\nabla v^*\|_{L_\infty}\big\|\partial_x^\alpha\sigma(t)\big\|^2 \\
&\quad + \sum_{\beta<\alpha}\big(\big\|\nabla\partial_x^{\alpha-\beta}v^*\big\|_{L_3}\big\|\partial_x^\beta\sigma(t)\big\|_{L_6} + \big\|\partial_x^{\alpha-\beta}v^*\big\|_{L_\infty}\big\|\nabla\partial_x^\beta\sigma(t)\big\|\big)\big\|\partial_x^\beta\sigma(t)\big\|\bigg\} \\
&\leqslant C\epsilon\big\|\nabla\sigma(t)\big\|_{|\alpha|-1}^2.
\end{aligned} \tag{3.11}$$

For K_2 and K_3, they can be estimated as follows:

$$\begin{aligned}
K_2 &\leqslant C\big\|\big(\nabla\rho^*, \nabla\sigma(t), \nabla\theta^*, \nabla\vartheta(t)\big)\big\|_{L_3}\big\|\nabla\partial_x^\alpha\vartheta(t)\big\|\big\|\partial_x^\alpha\vartheta(t)\big\|_{L_6} \\
&\leqslant C\epsilon\big\|\nabla\partial_x^\alpha\vartheta(t)\big\|^2,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
K_3 &\leqslant C\bigg\{\big\|\big(1+|x|\big)^2\big(\nabla\rho^*, \nabla\theta^*\big)\big\|_{L_\infty}\left\|\frac{\partial_x^\alpha w(t)}{|x|}\right\|\left\|\frac{\partial_x^\alpha\vartheta(t)}{|x|}\right\| \\
&\quad + \big\|(\nabla\sigma, \nabla\vartheta)(t)\big\|_{L_3}\big\|\partial_x^\alpha w(t)\big\|\big\|\partial_x^\alpha\vartheta(t)\big\|_{L_6}\bigg\} \\
&\leqslant C\epsilon\big\|\big(\nabla w(t), \nabla\vartheta(t)\big)\big\|_{|\alpha|,|\alpha|}.
\end{aligned} \tag{3.13}$$

Similar to that of K_2, K_4 can be estimated as follows

$$K_4 \leqslant C\epsilon\big\|\nabla\partial_x^\alpha w(t)\big\|^2. \tag{3.14}$$

To use Lemma 3.1, we divide K_5 into the following two parts

$$\begin{aligned}
K_5 &= \big|\big(F_\alpha(t), \tilde{A}(t)\partial_x^\alpha w(t)\big)\big| \\
&\quad + \left|\left(\frac{\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\big[\mu\Delta\partial_x^\alpha w(t) + (\mu+\mu')\nabla\big(\nabla\cdot\partial_x^\alpha w(t)\big)\big], \tilde{A}(t)\partial_x^\alpha w(t)\right)\right| \\
&\equiv K_{51} + K_{52}.
\end{aligned} \tag{3.15}$$

For K_{52}, by using integration by parts we have

$$\begin{aligned}
K_{52} &\leqslant \mu\left|\left(\nabla\left\{\frac{\tilde{A}(t)\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\partial_x^\alpha w(t)\right\}, \nabla\partial_x^\alpha w(t)\right)\right| \\
&\quad + (\mu+\mu')\left|\left(\nabla\cdot\left\{\frac{\tilde{A}(t)\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\partial_x^\alpha w(t)\right\}, \nabla\cdot\partial_x^\alpha w(t)\right)\right| \\
&\leqslant C\left\{\left\|\sigma(t)\right\|_{L_\infty}\left\|\nabla\partial_x^\alpha w(t)\right\|^2 + \left\|\left(\nabla\rho^*, \nabla\theta^*, \nabla\sigma(t), \nabla\vartheta(t)\right)\right\|_{L_3}\left\|\partial_x^\alpha w(t)\right\|_{L_6}\left\|\nabla\partial_x^\alpha w(t)\right\|\right\} \\
&\leqslant C\epsilon\left\|\nabla\partial_x^\alpha w(t)\right\|^2. \qquad (3.16)
\end{aligned}$$

To estimate K_{51}, we use (3.3). If $\alpha=0$,

$$\begin{aligned}
K_{51} &\leqslant C\left\{\left\|\left(1+|x|\right)^2\nabla v^*\right\|_{L_\infty}\left\|\frac{w(t)}{|x|}\right\|^2 + \left\|\left(1+|x|\right)v^*\right\|_{L_\infty}\right\}\left\|\nabla w(t)\right\|\left\|\frac{w(t)}{|x|}\right\| \\
&\quad + \left\|w(t)\right\|_{L_3}\left\|\nabla w(t)\right\|\left\|w(t)\right\|_{L_6} + \left\|\left(1+|x|\right)^2\left(\nabla\rho^*, \nabla\theta^*, \nabla^2 v^*\right)\right\|_{L_\infty}\left\|\frac{\sigma(t)}{|x|}\right\|\left\|\frac{w(t)}{|x|}\right\| \\
&\quad + \left\|\left(1+|x|\right)^2(\nabla\rho^*, \nabla\theta^*)\right\|_{L_\infty}\left\|\frac{\vartheta(t)}{|x|}\right\|\left\|\frac{w(t)}{|x|}\right\| \\
&\leqslant C\epsilon\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|^2, \qquad (3.17)
\end{aligned}$$

and if $1 \leqslant |\alpha| \leqslant 3$,

$$\begin{aligned}
K_{51} &\leqslant C\Bigg\{\left\|\nabla^{|\alpha|+1}v^*\right\|\left\|w(t)\right\|_{L_6}\left\|\nabla^{|\alpha|}w(t)\right\|_{L_3} + \sum_{\nu=1}^{|\alpha|+1}\left\|\nabla^\nu w(t)\right\|\left\|\nabla^{|\alpha|}w(t)\right\| \\
&\quad + \sum_{\nu=1}^{|\alpha|}\left\|\nabla^\nu\sigma(t)\right\|\left\|\nabla^{|\alpha|}w(t)\right\| + \sum_{\nu=1}^{|\alpha|}\left\|\nabla^\nu\vartheta(t)\right\|\left\|\nabla^{|\alpha|}w(t)\right\| \\
&\quad + \sum_{\nu=1}^{|\alpha|+1}\left\|\left(\nabla^\nu\rho^*, \nabla^\nu\theta^*, \nabla^{\nu+1}v^*\right)\right\|\left\|\sigma(t)\right\|_{L_6}\left\|\nabla^{|\alpha|}w(t)\right\|_{L_3} \\
&\quad + \sum_{\nu=1}^{|\alpha|+1}\left\|\left(\nabla^\nu\rho^*, \nabla^\nu\theta^*\right)\right\|\left\|\vartheta(t)\right\|_{L_6}\left\|\nabla^{|\alpha|}w(t)\right\|_{L_3} + \left\|R_F^{|\alpha|}(t)\right\|_{L_{3/2}}\left\|\nabla^{|\alpha|}w(t)\right\|_{L_3}\Bigg\} \\
&\leqslant C(\epsilon+\lambda)\left\|\left(\nabla\sigma, \nabla w, \nabla\vartheta\right)(t)\right\|^2_{|\alpha|-1,|\alpha|,|\alpha|-1} + C\lambda^{-1}\left\|\nabla^{|\alpha|}w(t)\right\|^2. \qquad (3.18)
\end{aligned}$$

Inserting (3.16), (3.18) into (3.15) deduce

$$K_5 \leqslant \begin{cases} C\epsilon\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\|^2, & \alpha=0, \\ C(\epsilon+\lambda)\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\|^2_{|\alpha|-1,|\alpha|,|\alpha|-1} + C\lambda^{-1}\|\nabla^{|\alpha|}w(t)\|^2, & 1\leqslant|\alpha|\leqslant 3. \end{cases} \qquad (3.19)$$

K_6 can be estimated in a way similar to that of K_5 by using (3.4). In fact if $\alpha=0$

$$K_6 \leqslant C\epsilon\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|^2, \qquad (3.20)$$

and if $1 \leqslant |\alpha| \leqslant 3$,

$$K_6 \leqslant C(\epsilon+\lambda)\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(t)\big\|^2_{|\alpha|-1,|\alpha|,|\alpha|}+C\lambda^{-1}\big\|\nabla^{|\alpha|}\vartheta(t)\big\|^2. \tag{3.21}$$

For $1 \leqslant |\alpha| \leqslant 3$, we can check that

$$K_7 \leqslant C\epsilon\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(t)\big\|^2_{|\alpha|-1,|\alpha|,|\alpha|}. \tag{3.22}$$

We note that the term $(I_\alpha(t),\nabla\partial_x^\alpha\sigma(t))$ is estimated by integration by parts, and for the case of $|\alpha|=3$ with the aid of the following inequality:

$$\left\|\frac{w(t)}{1+|x|}\right\|_{L_\infty} \leqslant C\big\|\nabla w(t)\big\|_1,$$

which follows from the Sobolev inequality and the Hardy inequality. In order to estimate K_8 and K_9, we use $(3.1)_1$ and $(3.1)_3$. In fact for K_8,

$$2K_8 \leqslant \big|\big(\tilde A_\rho(t)\sigma_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)\big|+\big|\big(\tilde A_\theta(t)\vartheta_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)\big| \equiv K_{81}+K_{82}. \tag{3.23}$$

Using $(3.1)_1$ and $(3.1)_3$, K_{81}, K_{82} can be estimated as follows respectively:

$$\begin{aligned}
K_{81} &= \big(\tilde A_\rho(t)\sigma_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)\\
&= \big|\big(\nabla\cdot\big\{\big(\rho^*+\sigma(t)\big)w(t)+\big(v^*\sigma(t)\big)\big\},\tilde A_\rho(t)\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\leqslant C\big|\big(\big(\rho^*+\sigma(t)\big)w(t)+v^*\sigma(t),\nabla\tilde A_\rho(t)\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)+2\tilde A_\rho(t)\nabla\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\leqslant C\big\{\big\|\big(w(t),\sigma(t)\big)\big\|_{L_6}\big\|\big(\nabla\rho^*,\nabla\theta^*,\nabla\sigma(t),\nabla\vartheta(t)\big)\big\|\,\big\|\partial_x^\alpha w(t)\big\|^2_{L_6}\\
&\qquad+\big(\big\|w(t)\big\|_{L_3}+\|v^*\|_{L_6}\big\|\sigma(t)\big\|_{L_6}\big)\big\|\nabla\partial_x^\alpha w(t)\big\|\,\big\|\partial_x^\alpha w(t)\big\|_{L_6}\big\}\\
&\leqslant C\epsilon\big\|\nabla\partial_x^\alpha w(t)\big\|^2, \qquad (3.24)\\
K_{82} &= \big(\tilde A_\theta(t)\vartheta_t(t)\partial_x^\alpha w(t),\partial_x^\alpha w(t)\big)\\
&= \big|\big(\kappa E(t)\Delta\big(\theta^*+\vartheta(t)\big)-D(t)\nabla\cdot\big(v^*+v(t)\big)-\big(v^*+v(t)\big)\cdot\nabla\big(\theta^*+\vartheta(t)\big)\\
&\qquad+E(t)\Psi\big(v^*+w(t)\big),\tilde A_\theta(t)\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&= \big|\big(\Delta\big(\theta^*+\vartheta(t)\big),\tilde A_\theta(t)\nabla\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\qquad+\big|\big(\nabla\big(v^*+w(t)\big),D(t)\tilde A_\theta(t)\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\qquad+\big|\big(-\big(v^*+v(t)\big)\nabla\big(\theta^*+\vartheta(t)\big),\tilde A_\theta(t)\nabla\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\qquad+\big|\big(\Psi\big(v^*+w(t)\big),E(t)\tilde A_\theta(t)\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\equiv K_{82}^1+\cdots+K_{82}^4. \qquad (3.25)
\end{aligned}$$

Using integrations by parts, we have

$$\begin{aligned}
K_{82}^1 &= \left|\left(\nabla\left(\theta^* + \vartheta(t)\right), \nabla\left(\kappa E(t)\tilde{A}_\theta(t)\right)\partial_x^\alpha w(t) \cdot \partial_x^\alpha w(t) + 2\kappa E(t)\tilde{A}_\theta(t)\nabla\partial_x^\alpha w(t) \cdot \partial_x^\alpha w(t)\right)\right| \\
&\leqslant C\left\{\left\|\left(\nabla\theta^*, \nabla\vartheta(t)\right)\right\|_{L_6}\left\|\left(\nabla\rho^*, \nabla\theta^*, \nabla\sigma(t), \nabla\vartheta(t)\right)\right\|\left\|\partial_x^\alpha w(t)\right\|_{L_6}^2\right. \\
&\quad \left. + \left\|\left(\nabla\theta^*, \nabla\vartheta(t)\right)\right\|_{L_3}\left\|\nabla\partial_x^\alpha w(t)\right\|\left\|\partial_x^\alpha w(t)\right\|_{L_6}\right\} \\
&\leqslant C\epsilon\left\|\nabla\partial_x^\alpha w(t)\right\|^2. \qquad (3.26)
\end{aligned}$$

Also, for the remaining terms, simple integration by parts for K_{82}^2 yields

$$K_{82}^k \leqslant C\epsilon\left\|\nabla\partial_x^\alpha w(t)\right\|^2, \qquad (3.27)$$

for $k = 2, 3, 4$. Consequently

$$K_8 \leqslant C\epsilon\left\|\nabla\partial_x^\alpha w(t)\right\|^2. \qquad (3.28)$$

At last, the term K_9 is estimated in a way similar to that of K_8, and we have

$$K_9 \leqslant C\epsilon\left\|\nabla\partial_x^\alpha \vartheta(t)\right\|^2. \qquad (3.29)$$

Combining (3.10)–(3.29), we can obtain (3.7) and (3.8), if we take $\alpha = \min_{(\rho,\theta)\in S(\bar{\rho},\bar{\theta})} \mu\frac{\tilde{A}(\rho,\theta)}{\rho}$, $\beta = \min_{(\rho,\theta)\in S(\bar{\rho},\bar{\theta})} \kappa E(\rho,\theta)\tilde{B}(\rho,\theta)$ and choose $\epsilon, \lambda > 0$ small enough. This completes the proof of Lemma 3.2. □

3.3. Estimates for $w_t(t)$, $\vartheta_t(t)$ and their derivatives up to $\nabla^2 w_t(t)$, $\nabla^2\vartheta_t(t)$

In this section, we give the estimates for $w_t(t)$, $\vartheta_t(t)$ and their derivatives up to $\nabla^2 w_t(t)$, $\nabla^2\vartheta_t(t)$. The main results are the following.

Lemma 3.3. *Let $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{3,3,3})$ be a solution to* (3.1). *Then there exist $\epsilon_0 > 0$, $\lambda_0 > 0$ and $\gamma > 0$ such that if $\epsilon \leqslant \epsilon_0$ and $\|(\sigma, w, \vartheta)(t)\|_{3,3,3}$, $\|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$, the following estimates hold*:

$$\frac{d}{dt}\left(\nabla\sigma(t), w(t)\right) + \gamma\left\|w_t(t)\right\|^2 \leqslant C\left\{\epsilon\left\|\nabla\sigma(t)\right\|^2 + \left\|\nabla w(t)\right\|_1^2 + \left\|\nabla\vartheta(t)\right\|^2\right\}, \qquad (3.30)$$

$$\begin{aligned}
&\frac{d}{dt}\left(\nabla^k\sigma(t), \nabla^{k-1}w(t)\right) + \gamma\left\|\nabla^{k-1}w_t(t)\right\|^2 \\
&\quad \leqslant C\left\|\left(\nabla\sigma, \nabla w, \nabla\vartheta\right)(t)\right\|_{k-2,k,k-2}^2 + C\epsilon\left\|\nabla^{k-2}w_t(t)\right\|^2, \qquad (3.31)
\end{aligned}$$

$$\left\|\vartheta_t(t)\right\|^2 \leqslant C\left\{\epsilon\left\|\nabla\sigma(t)\right\|^2 + \left\|\nabla w(t)\right\|^2 + \left\|\nabla\vartheta(t)\right\|_1^2\right\}, \qquad (3.32)$$

$$\left\|\nabla^{k-1}\vartheta_t(t)\right\|^2 \leqslant C\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|_{k-1,k-1,k}^2 \qquad (3.33)$$

for $2 \leqslant k \leqslant 3$, where $C > 0$ is a constant depending only on μ, μ', and κ.

Proof. Multiplying $(3.1)_2$ by $A(t)^{-1}$, we have

$$\frac{1}{A(t)}w_t(t)+\nabla\sigma(t)=\frac{1}{\rho^*A(t)}\big[\mu\Delta w(t)+(\mu+\mu')\nabla\big(\nabla\cdot w(t)\big)\big]-\frac{B(t)}{A(t)}\nabla\vartheta(t)+\frac{f(t)}{A(t)}.$$

For any multi-index α with $0\leqslant|\alpha|\leqslant 2$, applying ∂_x^α to this formula and multiplying the resultant equation by $\partial_x^\alpha w_t(t)$, we have

$$\begin{aligned}&\big(\nabla\cdot\partial_x^\alpha\sigma(t),\partial_x^\alpha w_t(t)\big)+\left(\frac{1}{A(t)}\partial_x^\alpha w_t(t),\partial_x^\alpha w_t(t)\right)\\&\quad=\left(\partial_x^\alpha\left\{\frac{1}{\rho^*A(t)}\big[\mu\Delta w(t)+(\mu+\mu')\nabla\big(\nabla\cdot w(t)\big)\big]-\frac{B(t)}{A(t)}\nabla\vartheta(t)+\frac{f(t)}{A(t)}\right\}\right.\\&\qquad\left.-I_\alpha(t),\partial_x^\alpha w_t(t)\right),\end{aligned}$$

where $I_\alpha(t)$ is defined by

$$I_\alpha(t)=\sum_{\beta<\alpha}\binom{\alpha}{\beta}\left(\partial_x^{\alpha-\beta}\frac{1}{A(t)}\right)\partial_x^\beta w_t(t).$$

The first term is written in the following form:

$$\big(\nabla\cdot\partial_x^\alpha\sigma(t),\partial_x^\alpha w_t(t)\big)=\frac{d}{dt}\big(\nabla\cdot\partial_x^\alpha\sigma(t),\partial_x^\alpha w(t)\big)+\big(\nabla\cdot\partial_x^\alpha\sigma_t(t),\partial_x^\alpha w(t)\big).$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}\big(\nabla\cdot\partial_x^\alpha\sigma(t),\partial_x^\alpha w(t)\big)+\left(\frac{1}{A(t)}\partial_x^\alpha w_t(t),\partial_x^\alpha w_t(t)\right)\\&\quad\leqslant\left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^*A(t)}\big[\mu\Delta w(t)+(\mu+\mu')\nabla\big(\nabla\cdot w(t)\big)\big]\right\},\partial_x^\alpha w_t(t)\right)\right|+\big|\big(I_\alpha(t),\partial_x^\alpha w_t(t)\big)\big|\\&\qquad+\left|\left(\partial_x^\alpha\left\{\frac{f(t)}{A(t)}\right\},\partial_x^\alpha w_t(t)\right)\right|+\left|\left(\partial_x^\alpha\left\{\frac{B(t)}{A(t)}\nabla\vartheta(t)\right\},\partial_x^\alpha w_t(t)\right)\right|+\big|\big(\nabla\cdot\partial_x^\alpha\sigma_t(t),\partial_x^\alpha w(t)\big)\big|\\&\quad\equiv K_1+\cdots+K_5.\end{aligned}\tag{3.34}$$

Now we estimate the right-hand side of (3.34) term by term by using the Sobolev inequality. Firstly, it is easily to check that

$$K_1\leqslant\lambda\big\|\nabla^{|\alpha|}w_t(t)\big\|^2+C\lambda^{-1}\big\|\nabla^2w(t)\big\|_{|\alpha|}^2,\tag{3.35}$$

$$K_2\leqslant C\epsilon\big\|\nabla w_t(t)\big\|_{|\alpha|-1}^2.\tag{3.36}$$

To estimate K_3, if $\alpha=0$, we have from Lemma 3.1 that $\alpha=0$,

$$\begin{aligned}K_3 &\leqslant \left(\frac{1}{A(t)}\bigl\{|\nabla v^*|\bigl|w(t)\bigr| + \bigl(|v^*| + \bigl|w(t)\bigr|\bigr)\bigl|\nabla w(t)\bigr| + \bigl(|\nabla\rho^*| + |\nabla\theta^*| + \bigl|\nabla^2 v^*\bigr|\bigr)\bigl|\sigma(t)\bigr|\right.\\&\quad\left.+ \bigl(|\nabla\rho^*| + |\nabla\theta^*|\bigr)\bigl|\vartheta(t)\bigr|\bigr\}, \bigl|w_t(t)\bigr|\right)\\&\leqslant C\bigl\{\|\nabla v^*\|_{L_3}\bigl\|w(t)\bigr\|_{L_6} + \bigl\|\bigl(v^*, w(t)\bigr)\bigr\|_{L_\infty}\bigl\|\nabla w(t)\bigr\|\\&\quad+ \bigl\|\bigl(\nabla\rho^*, \nabla\theta^*, \nabla^2 v^*\bigr)\bigr\|_{L_3}\bigl\|\sigma(t)\bigr\|_{L_6} + \bigl\|(\nabla\rho^*, \nabla\theta^*)\bigr\|\bigl\|\vartheta(t)\bigr\|_{L_6}\bigr\}\bigl\|w_t(t)\bigr\|\\&\leqslant C\epsilon\bigl\{\bigl\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\bigr\|_{0,1,0}^2 + \bigl\|w_t(t)\bigr\|^2\bigr\}.\end{aligned}\tag{3.37}$$

If $1 \leqslant \alpha \leqslant 2$, due to

$$\begin{aligned}K_3 &\leqslant \sum_{\beta<\alpha}\binom{\alpha}{\beta}\bigl|\bigl(\bigl(\partial_x^{\alpha-\beta}A(t)^{-1}\bigr)\partial_x^\beta f(t), \partial_x^\alpha w_t(t)\bigr)\bigr| + \bigl|\bigl(A(t)^{-1}\partial_x^\alpha f(t), \partial_x^\alpha w_t(t)\bigr)\bigr|\\&\equiv K_{31} + K_{32},\end{aligned}\tag{3.38}$$

we have

$$\begin{aligned}K_{31} &\leqslant C\sum_{\nu=1}^{|\alpha|}\bigl\|\bigl(\nabla\rho^*, \nabla\theta^*, \nabla\sigma(t), \nabla\vartheta(t)\bigr)\bigr\|_{L_3}\\&\quad\times\sum_{\substack{0\leqslant\nu_1\leqslant|\alpha|-1\\0\leqslant\nu_2\leqslant|\alpha|+1}}\bigl\|\bigl(\nabla^{\nu_1}\sigma, \nabla^{\nu_2}w, \nabla^{\nu_1}\vartheta\bigr)(t)\bigr\|_{L_6}\bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|\\&\leqslant C\epsilon\bigl\{\bigl\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\bigr\|_{|\alpha|-1,|\alpha|+1,|\alpha|-1}^2 + \bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|^2\bigr\}\end{aligned}\tag{3.39}$$

and

$$\begin{aligned}K_{32} &\leqslant C\left\{\|\nabla v^*\|_{L_3}\bigl\|w(t)\bigr\|_{L_6} + \sum_{\nu=1}^{|\alpha|+1}\bigl\|\bigl(\nabla^\nu\rho^*, \nabla^{\nu+1}v^*\bigr)\bigr\|_{L_3}\bigl\|\sigma(t)\bigr\|_{L_6}\right.\\&\quad\left.+ \bigl\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\bigr\|_{|\alpha|-1,|\alpha|+1,|\alpha|-1} + \bigl\|R_F^{|\alpha|}(t)\bigr\|\right\}\bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|\\&\leqslant \lambda\bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|^2 + C\lambda^{-1}\bigl\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\bigr\|_{|\alpha|-1,|\alpha|+1,|\alpha|-1}^2.\end{aligned}\tag{3.40}$$

Thus for $0 \leqslant |\alpha| \leqslant 2$

$$K_3 \leqslant (C\epsilon + \lambda)\bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|^2 + C\bigl(\epsilon + \lambda^{-1}\bigr)\bigl\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\bigr\|_{|\alpha|-1,|\alpha|+1,|\alpha|-1}^2.\tag{3.41}$$

For K_4, we can get the following estimate by employing a similar argument

$$K_4 \leqslant \lambda\bigl\|\nabla^{|\alpha|}w_t(t)\bigr\|^2 + C\lambda^{-1}\bigl\|\nabla\vartheta(t)\bigr\|_{|\alpha|}^2.\tag{3.42}$$

At last, to deduce an estimate on K_5, we substitute $(3.1)_1$ into σ_t as in (3.24). Indeed, if $\alpha = 0$,

$$
\begin{aligned}
K_5 &\leqslant \big|\big(\nabla\cdot\big\{\big(\rho^*+\sigma(t)\big)w(t)\big\},\nabla\big(\nabla\cdot w(t)\big)\big)\big|+\big|\big(v^*\sigma(t),\nabla\cdot w(t)\big)\big| \\
&\leqslant C\bigg\{\big\|\big(\nabla\rho^*,\nabla\sigma(t)\big)\big\|_{L_3}\big\|w(t)\big\|_{L_6}\big\|\nabla w(t)\big\|+\big\|\nabla w(t)\big\|^2 \\
&\quad+\big\|\big(1+|x|\big)v^*\big\|_{L_\infty}\bigg\|\frac{\sigma(t)}{|x|}\bigg\|\big\|\nabla^2 w(t)\big\|\bigg\} \\
&\leqslant C\epsilon\big\|\big(\nabla\sigma,\nabla^2 w\big)(t)\big\|^2+C\big\|\nabla w(t)\big\|^2,
\end{aligned}
\tag{3.43}
$$

and if $1\leqslant|\alpha|\leqslant 2$,

$$
\begin{aligned}
K_5 &\leqslant \big|\big(\partial_x^\alpha\big\{\big(\rho^*+\sigma(t)\big)w(t)+v^*\sigma(t)\big\},\nabla\big(\nabla\cdot w(t)\big)\big)\big| \\
&\leqslant C\sum_{\beta<\alpha}\big\{\big\|\big(\partial_x^{\alpha-\beta}\rho^*,\partial_x^{\alpha-\beta}\sigma(t)\big)\big\|_{L_3}\big\|\partial_x^\beta w(t)\big\|_{L_6}+\big\|\partial_x^\alpha w(t)\big\| \\
&\quad+\big\|\partial_x^{\alpha-\beta}v^*\big\|_{L_3}\big\|\partial_x^\beta\sigma(t)\big\|_{L_6}+\big\|\partial_x^\beta\sigma(t)\big\|\big\}\big\|\nabla^2\partial_x^\alpha w(t)\big\| \\
&\leqslant C\epsilon\big\|(\nabla\sigma,\nabla w)(t)\big\|^2_{|\alpha|-1,|\alpha|+1}.
\end{aligned}
\tag{3.44}
$$

Combining (3.34)–(3.44), we can obtain (3.30) and (3.31) if we take $\gamma=\min_{(\rho,\theta)\in S(\bar\rho,\bar\theta)}\frac{1}{2A(\rho,\theta)}$ and ϵ, λ small enough.

For any multi-index α with $0\leqslant|\alpha|\leqslant 2$, applying ∂_x^α to $(3.1)_3$ and multiplying the resultant equation by $\partial_x^\alpha\vartheta_t(t)$, we have

$$
\begin{aligned}
\big\|\partial_x^\alpha\vartheta_t(t)\big\|^2 &\leqslant \big|\big(\kappa\partial_x^\alpha\big(E^*\Delta\vartheta(t)\big),\partial_x^\alpha\vartheta_t(t)\big)\big|+\big|\big(\partial_x^\alpha\big(D(t)\Delta w(t)\big),\partial_x^\alpha\vartheta_t(t)\big)\big| \\
&\quad+\big|\big(\partial_x^\alpha h(t),\partial_x^\alpha\vartheta_t(t)\big)\big| \\
&\equiv K_6+K_7+K_8.
\end{aligned}
\tag{3.45}
$$

The terms in the right-hand side of (3.45) can be estimated as follows

$$
K_6\leqslant\lambda\big\|\nabla^{|\alpha|}\vartheta_t(t)\big\|^2+C\lambda^{-1}\big\|\nabla^2\vartheta(t)\big\|^2_{|\alpha|}.
$$

If $\alpha=0$,

$$
\begin{aligned}
K_7&\leqslant\lambda\big\|\vartheta_t(t)\big\|^2+C\lambda^{-1}\big\|\nabla w(t)\big\|^2,\\
K_8&\leqslant C\epsilon\big\{\big\|\vartheta_t(t)\big\|^2+\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(t)\big\|^2_{0,0,1}\big\},
\end{aligned}
$$

and if $1\leqslant|\alpha|\leqslant 2$,

$$
\begin{aligned}
K_7&\leqslant\lambda\big\|\nabla^{|\alpha|}\vartheta_t(t)\big\|^2+C\lambda^{-1}\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(t)\big\|^2_{|\alpha|-1,|\alpha|,|\alpha|+1},\\
K_8&\leqslant\lambda\big\|\nabla^{|\alpha|}\vartheta_t(t)\big\|^2+C\lambda^{-1}\big\|(\nabla\sigma,\nabla w,\nabla\vartheta)(t)\big\|^2_{|\alpha|,|\alpha|,|\alpha|}+C\epsilon\big\|\nabla^{|\alpha|+2}\vartheta(t)\big\|^2.
\end{aligned}
$$

Combining the above estimates with (3.45), we can get (3.32), (3.33) if taking $\epsilon,\lambda>0$ small enough. This completes the proof of Lemma 3.3. □

3.4. Estimates for $\nabla\sigma(t)$ and its derivatives up to $\nabla^3\sigma(t)$

Finally, we also have the following estimates for $\nabla\sigma(t)$ and its derivatives up to $\nabla^3\sigma(t)$.

Lemma 3.4. *Let $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{3,3,3})$ be a solution to (3.1). Then there exist $\epsilon_0 > 0$ and $\lambda_0 > 0$ such that if $\epsilon \leqslant \epsilon_0$ and $\|(\sigma, w, \vartheta)(t)\|_{3,3,3}, \|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$, the following estimates hold*:

$$\left\|\nabla\sigma(t)\right\|^2 \leqslant C\left\|\left(\nabla w, \nabla\vartheta, w_t(t)\right)\right\|^2_{1,0,0}, \tag{3.46}$$

$$\left\|\nabla^k\sigma(t)\right\|^2 \leqslant C\left\{\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|^2_{k-2,k,k-2} + \left\|\nabla^{k-1}w_t(t)\right\|^2\right\} \tag{3.47}$$

for $2 \leqslant k \leqslant 3$, where $C > 0$ is a constant depending only on μ, μ', and κ.

Proof. For any multi-index α with $0 \leqslant |\alpha| \leqslant 2$, applying ∂_x^α to $(3.1)_2$ and multiplying the resultant equation by $\partial_x^\alpha\sigma_t(t)$, we have

$$\begin{aligned}
&\left(A(t)\nabla\partial_x^\alpha\sigma(t), \nabla\partial_x^\alpha\sigma(t)\right)\\
&\quad\leqslant \left|\left(\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t)\right)\right|\\
&\qquad+\left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^*}\left[\mu\Delta w(t) + (\mu+\mu')\nabla\left(\nabla\cdot w(t)\right)\right]\right\}, \nabla\partial_x^\alpha\sigma(t)\right)\right|\\
&\qquad+\left|\left(\partial_x^\alpha\left(B(t)\nabla\vartheta(t)\right), \nabla\partial_x^\alpha\sigma(t)\right)\right|\\
&\qquad+\left|\left(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\right)\right| + \left|\left(\partial_x^\alpha f(t), \nabla\partial_x^\alpha\sigma(t)\right)\right|\\
&\quad\equiv K_1 + \cdots + K_5,
\end{aligned} \tag{3.48}$$

where $I_\alpha(t)$ is defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\left(\partial_x^{\alpha-\beta}A(t)\right)\nabla\partial_x^\alpha\sigma(t).$$

Employing Lemma 3.1, it follows from the Sobolev inequality that

$$\begin{aligned}
&K_1 \leqslant \lambda\left\|\nabla^{|\alpha|+1}\sigma(t)\right\|^2 + C\lambda^{-1}\left\|\nabla^{|\alpha|}w_t(t)\right\|^2,\\
&K_2 \leqslant \lambda\left\|\nabla^{|\alpha|+1}\sigma(t)\right\|^2 + C\lambda^{-1}\left\|\nabla^2 w(t)\right\|^2_{|\alpha|},\\
&K_3 \leqslant \lambda\left\|\nabla^{|\alpha|+1}\sigma(t)\right\|^2 + C\lambda^{-1}\left\|\nabla^2\vartheta(t)\right\|^2_{|\alpha|},\\
&K_4 \leqslant C\epsilon\left\|\nabla\sigma(t)\right\|^2_{|\alpha|},\\
&K_5 \leqslant \lambda\left\|\nabla^{|\alpha|+1}\sigma(t)\right\|^2 + C\lambda^{-1}\left\|(\nabla\sigma, \nabla w, \nabla\vartheta)(t)\right\|^2_{|\alpha|-1,|\alpha|+1,|\alpha|-1}.
\end{aligned}$$

Here, if $\alpha = 0$, $\|\nabla\sigma(t)\|_{-1}$ is thought to be zero. Combining the above estimates with (3.48), we obtain (3.46)–(3.47) if we can take ϵ, λ small enough. This completes the proof of Lemma 3.4. $\square$

3.5. Proof of Proposition 3.2

Let $(\sigma, w, \vartheta)(t) \in \mathcal{C}(0, t_1; \mathcal{H}^{3,3,3})$ be a solution to (3.1) locally in time. Furthermore, we suppose that $\|(\sigma, w, \vartheta)(t)\|_{3,3,3}, \|(\sigma^*, v^*, \vartheta^*)\|_{\mathcal{J}^{4,5,5}} \leqslant \epsilon$, where $\epsilon > 0$ is small enough such that at least we can use the results obtained in Lemmas 3.2–3.4. Also we use the notation:

$$[\sigma, w, \vartheta](t) = \|\sigma(t)\|^2 + \big(\tilde{A}(t)w(t), w(t)\big) + \big(\tilde{B}(t)\vartheta(t), \vartheta(t)\big),$$

where $\tilde{A}$, $\tilde{B}$ are defined as in Lemma 3.2.

Summing up (3.7), (3.8) with $k = 1$, (3.30), (3.32) and (3.46) (after multiplying (3.8), (3.30), (3.32), (3.46) with small numbers respectively), we have

$$\frac{d}{dt}\left\{\sum_{\nu=0}^{1}\alpha_\nu\big[\nabla^\nu\sigma, \nabla^\nu w, \nabla^\nu\vartheta\big] + \beta_1(\nabla\sigma, w)\right\} + \|(\nabla\sigma, \nabla w, \nabla\vartheta)\|_{0,1,1}^2 + \|(w_t, \vartheta_t)\|^2 \leqslant 0 \tag{3.49}$$

if we take $\epsilon, \lambda > 0$ sufficiently small. Here and hereafter $\alpha_k, \beta_k > 0$ are constants depending only on μ, μ' and κ. Then summing up (3.8), (3.31), (3.33), (3.47) with $k = 2$ and (3.49), we have

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{\nu=0}^{2}\alpha_\nu\big[\nabla^\nu\sigma, \nabla^\nu w, \nabla^\nu\vartheta\big] + \sum_{\nu=1}^{2}\beta_\nu\big(\nabla^\nu\sigma, \nabla^{\nu-1}w\big)\right\}\\&\quad + \|(\nabla\sigma, \nabla w, \nabla\vartheta)\|_{1,2,2}^2 + \|(w_t, \vartheta_t)\|_{1,1}^2 \leqslant 0.\end{aligned} \tag{3.50}$$

Also, by (3.8), (3.31), (3.33), (3.47) with $k = 3$ and (3.50), we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{\nu=0}^{3}\alpha_\nu\big[\nabla^\nu\sigma, \nabla^\nu w, \nabla^\nu\vartheta\big] + \sum_{\nu=1}^{3}\beta_\nu\big(\nabla^\nu\sigma, \nabla^{\nu-1}w\big)\right\}\\&\quad + \|(\nabla\sigma, \nabla w, \nabla\vartheta)\|_{2,3,3}^2 + \|(w_t, \vartheta_t)\|_{2,2}^2 \leqslant 0\end{aligned} \tag{3.51}$$

for any $t \in [0, t_1]$. Then integrating of (3.51) with respect to t over $[0, t]$ implies that

$$N[\sigma, w, \vartheta](t) + \int_0^t \|(\nabla\sigma, \nabla w, \nabla\vartheta)(s)\|_{2,3,3}^2 + \|(w_t, \vartheta_t)(s)\|_{2,2}^2\, ds \leqslant N[\sigma, w, \vartheta](0), \tag{3.52}$$

where $N[\sigma, w, \vartheta](s)$ is defined by

$$N[\sigma, w, \vartheta](s) \equiv \sum_{\nu=0}^{3}\alpha_\nu\big[\nabla^\nu\sigma, \nabla^\nu w, \nabla^\nu\vartheta\big](s) + \sum_{\nu=1}^{3}\beta_\nu\big(\nabla^\nu\sigma(s), \nabla^{\nu-1}w(s)\big)$$

for any $s > 0$.

Let us denote $B_0 = \min_{(\rho,\theta)\in S(\bar\rho,\bar\theta)}\{\tilde A(\rho,\theta), \tilde B(\rho,\theta), 1\}$ and $B_1 = \max_{(\rho,\theta)\in S(\bar\rho,\bar\theta)}\{\tilde A(\rho,\theta), \tilde B(\rho,\theta), 1\}$. Since we may assume without loss of generality that $\alpha_k \leqslant \alpha_{k-1}$ and $\beta_k \leqslant \alpha_k \min\{B_0, 1\}/4$ for $k = 1, 2, 3$, it follows from simple calculation that

$$\frac{\alpha_3}{4} B_0 \left\| (\sigma, w, \vartheta)(s) \right\|_{3,3,3}^2 \leqslant N[\sigma, w, \vartheta](s) \leqslant 2\alpha_0 B_1 \left\| (\sigma, w, \vartheta)(s) \right\|_{3,3,3}^2 \tag{3.53}$$

for each $s \in [0, t_1]$. Combining (3.52) and (3.53), we obtain (3.2), which completes the proof of Proposition 3.2.

Hence, by Propositions 3.1 and 3.2, we finally arrive at the conclusion of Theorem 1.2.

References

[1] K. Deckelnick, Decay estimates for the compressible Navier–Stokes equations in unbounded domains, Math. Z. 209 (1992) 115–130.

[2] R.-J. Duan, S. Ukai, T. Yang, H.-J. Zhao, Optimal convergence rates for the compressible Navier–Stokes equations with potential forces, Math. Methods Appl. Sci.

[3] D. Hoff, K. Zumbrun, Multidimensional diffusion waves for the Navier–Stokes equations of compressible flow, Indiana Univ. Math. J. 44 (1995) 604–676.

[4] Y. Kagei, T. Kobayashi, Asymptotic behavior of solutions of the compressible Navier–Stokes equations on the half space, Arch. Ration. Mech. Anal. 177 (2005) 231–330.

[5] T. Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in $\mathbb{R}^3$, J. Differential Equations 184 (2002) 587–619.

[6] T. Kobayashi, Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbf{R}^3$, Comm. Math. Phys. 200 (1999) 621–659.

[7] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2: Compressible Models, Oxford Univ. Press, 1998.

[8] T.-P. Liu, W.-K. Wang, The pointwise estimates of diffusion waves for the Navier–Stokes equations in odd multi-dimensions, Comm. Math. Phys. 196 (1998) 145–173.

[9] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979) 337–342.

[10] A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980) 67–104.

[11] A. Matsumura, T. Nishida, Initial–boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Comm. Math. Phys. 89 (4) (1983) 445–464.

[12] A. Matsumura, T. Nishida, Exterior stationary problems for the equation of motion of compressible viscous and heat-conductive fluids, in: Differential Equations, Xanthi, 1987, in: Lect. Notes Pure Appl. Math., vol. 118, Dekker, New York, 1989, pp. 473–479.

[13] A. Novotny, M. Padula, Physically reasonable solutions to steady compressible Navier–Stokes equations in $3D$-exterior domains ($v_\infty = 0$), J. Math. Kyoto Univ. 36 (1996) 389–422.

[14] A. Novotny, M. Padula, Physically reasonable solutions to steady compressible Navier–Stokes equations in $3D$-exterior domains ($v_\infty \neq 0$), Math. Ann. 308 (1997) 439–489.

[15] A.V. Novotny, I. Straskraba, Introduction to the Mathematical Theory of Compressible Flow, Oxford Univ. Press, 2004.

[16] G. Ponce, Global existence of small solutions to a class of nonlinear evolution equations, Nonlinear Anal. 9 (1985) 339–418.

[17] Y. Shibata, K. Tanaka, On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance, J. Math. Soc. Japan 55 (2003) 797–826.

[18] Y. Shibata, K. Tanaka, Rate of convergence of non-stationary flow to the steady flow of compressible viscous fluid, preprint.

[19] K. Tanaka, A stability of steady flow of compressible viscous fluid with respect to initial disturbance ($v_\infty \neq 0$), Math. Methods Appl. Sci.

[20] K. Tanaka, Rate of convergence of non-stationary flow to the steady flow of compressible viscous fluid, preprint.

[21] S. Ukai, T. Yang, H.-J. Zhao, Convergence rate for the compressible Navier–Stokes equations with external force, J. Hyperbolic Differ. Equ. 3 (3) (2006) 561–574.